HEAT KERNAL ANALYSIS ON WEIGHTED DIRICHLET SPACES

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

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May 2011

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Cornell University 2011

This thesis is concerned with heat kernel estimates on weighted Dirichlet spaces. The Dirichlet forms considered here are strongly local and regular. They are defined on a complete locally compact separable metric space. The associated heat equation is similar to that of local divergence form differential operators. The weight functions studied have the form of a function of the distance from a closed set Σ, that is, $x \mapsto a(d(x, \Sigma))$. We place conditions on the geometry of the set Σ and the growth rate of function a itself. The function a can either blow up at 0 or ∞ or both. Some results include the case where Σ separates the whole spaces. It can also apply to the case where Σ do not separate the space, for example, a domain Ω and its boundary $\Sigma = \partial\Omega$. The condition on Σ is rather mild and do not assume differentiability condition.

BIOGRAPHICAL SKETCH

Santi Tasena was born in Chaing Rai, Thailand. He has been interests in mathematics since young. Perhaps since he has been forced to help out with the family's store. Counting changes everyday is what triggered him into loving numbers, and perhaps inspired him into study mathematics later. Not what his parents had in mind though.

After spending three years in high school being exposed to many subjects, he surprised everyone by accepting a scholarship from DPST(Development and Promotion of Science and Technology Talents Project, Thailand) to pursue his studied in mathematics. Four years later everyone was struck again by his decision to study aboard. Little did everyone expects a guy with such lousy English skills would be able to study in America.

After graduation, he will be going back to Thailand and perhaps live a simple life over there. Hopefully, there will be no more surprises this time around.

For my parents and family.

ACKNOWLEDGEMENTS

I would like to thank my advisor, Laurent Saloff-Coste, who introduce me to this interesting and challenging problem. It was so much fun working in this area even though I only understand a fraction of it. Much is left to explore after graduation. Nevertheless, this work would not have happenned without his patient, support and guidance. I would also like to thank the member of my committee, Leonard Gross and Xiaodong Cao for their contributions throughout the years. I owe most of my understanding of the subjects to the three of them.

The experience in Ithaca has been really wonderful and I must thank the Math department and DPST for giving me a chance to be here. Without these two institutions, I would not be here enjoying mathematics and meeting all those wonderful people, both mathematicians and non-mathematicians.

I also would like to thank all the people I have encountered all these years: all my colleagues and friends, all my teachers and instructors. Our encounters is what have shaped me into who I am. Lastly, I thank my parents. What they have done for me are beyond words and comparison.

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CHAPTER 1

INTRODUCTION

This chapter reviews some history of heat kernel estimates as it is related to the work done in this thesis. Even though the subject is less than sixty years old, it has progressed in many directions – [6],[31],[39],[41],[18],[15],[43],[42],[49]. For example, readers interested in Riemannian manifolds should look at Saloff-Coste's book[39] or Grigor'yan's recent book[15] while those interested in Lie group should have a look at Varopoulos[48]. Delmotte[7] also study heat equation on graphs. Recently, Ohta and Sturm[37] also study heat equation on Finsler manifolds. Several reviews of the subject are also available[40, 1, 38, 24, 12, 13, 11, 22, 14]. It is impossible to cover all aspect of the subject. What is written here is rather incomplete and the author apologies for any missing work that is not mentioned.

The classical heat equation is given by

$$\begin{aligned} \partial_t u &= \sum_{i=1}^{n} \Big(\frac{\partial}{\partial x_i}\Big)^2 u \\ u(0,\cdot) &= f \end{aligned}$$

for any appropriate function $f : \mathbb{R}^n \to \mathbb{R}$. The solution u of the heat equation represents the evolution of temperature over time: $u(t,x)$ is the temperature at point x at time $t \geq 0$ given the initial temperature distribution f.

It is well-known that the solution u is given by

$$u(t,x) = \int_{\mathbb{R}^n} f(y) \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/4t} dy$$

The function $p(t,x,y) = \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/4t}$ is called the heat kernel associated to the Laplacian $\sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)^2$. One important characteristics of $p(t,x,y)$ is that it depends only on the distance $d(x,y) = |x-y|$, not x and y directly. As time $t \to \infty$, the distances $d_t(x,y) = \sqrt{\frac{|x-y|^2}{4t}}$ are getting smaller. This corresponding to the fact heat spreads further and further away as time goes by. Another important characteristics is the term $t^{n/2}$. It might not be cleared at the moment what it is related to. This could come from the dimension constant n, or the volume of a ball of radius t(which is equal to t^n).

In general, the heat equation is given by

$$\begin{aligned} \partial_t u &= Lu \\ u(0,\cdot) &= f \end{aligned}$$

for any appropriate function $f : X \to \mathbb{R}$. Here L is an infinitesimal operator associated to a Markov semigroup $P_t = e^{-Lt}$. For example, the Laplacian in the classical heat equation mensioned earlier is the infinitesimal operator associated to the Brownian semigroup. Another example is $L = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j}$ where $a_{ij} = a_{ji}$ are smooth functions on $\mathbb{R}^n$. Nash, Moser and Aronson[32, 28, 29, 30, 3, 4, 5] shows that if (a_{ij}) is uniformly elliptic i.e. there exists a constant $c \geq 1$ such that

$$c^{-1} \sum_{i,j} \xi_i \xi_j \leq \sum_{i,j} a_{ij} \xi_i \xi_j \leq c \sum_{i,j} \xi_i \xi_j$$

for all $(\xi_i) \in \mathbb{R}^n$, then the heat kernel associated to $L = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j}$ satisfies the heat kernel estimates

$$\frac{c_1}{t^{n/2}} e^{-|x-y|^2/c_2 t} \leq p(t,x,y) \leq \frac{c_3}{t^{n/2}} e^{-|x-y|^2/c_4 t}$$

for some fixed constants $c_1, \ldots, c_4$. Their approach is based on Harnack inequality. Nash's approach is more primitive[9, 32] while Moser's approach is easier to generalize[28, 29, 30]. It is well-known today as Moser iteration method. The Harnack inequality is in itself an interesting subject. Readers interested in the history of Harnack inequality may look at Kassmann's article[21].

In 1986, P. Li and S.T.Yau[23] prove the following heat kernel estimates

$$\frac{c_1}{\sqrt{vol(x,\sqrt{t})vol(y,\sqrt{t})}}e^{-c_2 d(x,y)^2/t} \leq p(t,x,y) \leq \frac{c_3}{\sqrt{vol(x,\sqrt{t})vol(y,\sqrt{t})}}e^{-c_4 d(x,y)^2/t}$$

for the Laplacian on Riemannian manifolds with nonnegative Ricci curvature. The proof is based on S.T.Yau earlier work on Harnack inequality. For detail proof, the author suggests interested readers to take a look at R. Schoen and S.T.Yau's book[41].

The heat kernel estimates in this case differs from the Euclidean case. The term $t^{n/2}$ is replaced by $\sqrt{vol(x,\sqrt{t})vol(y,\sqrt{t})}$. Of course, in Euclidean space, both terms differ only by a constant multiple. Riemannian manifolds on the other hands, one must take into account the inhomogeneity of the volume measure.

Saloff-Coste[39] and many others take a functional analytic approach in proving heat kernel estimates and arrive at equivalent conditions, one of which is parabolic Harnack inequality. The results is then further extended to the general setting, strongly local regular Dirichlet spaces, by K.T.Sturm[45, 46, 47].

In this work, the author proves heat kernel estimates on weighted Dirichlet spaces. This could be seen as a perturbation result. But the "perturbation" can be very significant. The basic setting is as follows. First, one starts with a strongly local, regular, Dirichlet form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ on a L^2-space of a complete, locally compact, separable topological space. The Dirichlet form can be given in the form

$$\mathcal{E}(f,g) = \int d\Gamma(f,g)$$

for all $f,g \in \mathcal{D}(\mathcal{E})$. The weighted Dirichlet space corresponding to a weight h is given by

$$\mathcal{E}^h(f,g) = \int h d\Gamma(f,g)$$

for any f, g in an appropriate domain $\mathcal{D}(\mathcal{E}^h)$. The detailed construction of this is given in Chapter 6. Assuming that the original Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ satisfies the parabolic Harnack inequality, we consider the question of finding assumptions that imply that the weighted Dirichlet form $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ also satisfies the parabolic Harnack inequality.

If the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ corresponds to the Laplacian in a Riemannian manifold or Euclidean spaces, the result can be extended to a local divergence form differential operators $L = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j}$ when (a_{ij}) are comparable to h in the following sense

$$c^{-1} h(\xi) \sum_{i,j} \xi_i \xi_j \leq \sum_{i,j} a_{ij} \xi_i \xi_j \leq c h(\xi) \sum_{i,j} \xi_i \xi_j, \qquad \forall \xi = (\xi_i)$$

This could be seen as a generalization of Nash and Moser's results.

Actually this work is not the first studying weighted Dirichlet spaces. Saloff-Coste and Grigor'yan[10] also study such results on weighted manifolds. However, their results do not allow h to be infinite or vanish even though one would easily apply their proof to the case where h has unique singularity i.e. $\{h = \infty\}$ is a singleton.

Moschini and Tesei[27] also prove results on weighted Euclidean space $\mathbb{R}^n$, $n > 1$ for the weight $h(x) = |x|^{-k}$, $k < n$. All of these results relies on the fact that the complement of singularity set $\{h = \infty\}$ is connected. In this thesis, the author try to extend the result to the case where the singularity set separates the space. The weighted function, on another hand, is assumed to take the simple form of $h(x) = a(d(x, \Sigma))$ where Σ is the singularity set. At the end, the author obtain similar result by assume the following conditions.

The function $a : [0, \infty) \to (0, \infty]$ is assumed to be remotely constant i.e. there exists a constant $c \geq 1$ such that

$$\sup_{[r,3r]} a \leq c \inf_{[r,3r]} a < \infty$$

for all $r > 0$. This means that the function a is uniformly comparable to constant on balls far away to 0. Any weight function with unique singularity in any doubling space is equivalent to a function in this form. This is not true in general, however.

A closed set Σ is assumed to be ρ-accessible i.e. it has measure zero, satisfies the ρ'-skew condition for some $\rho' > \rho$ and that the cone $\{x : \rho d(x, o) \leq d(x, \Sigma) \leq r\}$ is connected for all $o \in \Sigma$ and $r > 0$. Examples of ρ-accessible sets include closed subsets of hyperplane in Euclidean spaces, boundary of uniform domains and of Reifenberg domains.

One of the results the author obtain is the following theorem.

Theorem 1.0.1 *Let $(\mathcal{E}, \mathcal{D}(E))$ be a strongly local, regular, Dirichlet form in $L^2(X, \nu)$ with intrinsic metric d satisfies the usual assumptions. Let also $\Sigma \subset X$ be a ρ-accessible subset, $d\mu = hd\nu$ where $h(x) = a(d(x, \Sigma))$, and $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ be the weighted Dirichlet form corresponding to function h.*

Assume $(\mathcal{E}, \mathcal{D}(E))$ satisfies the parabolic Harnack inequality and $a > 0$ on X is nonincreasing, and remotely constant. Then the Parabolic Harnack inequality for $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ holds for all balls whenever μ is doubling.

The author also gives sufficient condition for the weight measure $d\mu = a(d(\cdot, \Sigma))d\nu$ to be doubling. The result works best when ν is Alfor-regular of dimension α_ν and Σ is a self-similar fractal of dimension α_Σ. In this case it can be proved that μ is doubling if $1/a = o(r^{\alpha_\nu - \alpha_\Sigma})$. This is exactly what happen when Σ

is a hyperplane in Euclidean space.

Another related work is that of D.W. Robinson and his colleagues(e.g. [2, 35, 36]). Heat kernels asociated to degenerate operators were studied in their works. Heat kernels upper bounded were given, including the case that zero sets separated spaces. Their approach and assumptions, however, are different from this thesis.

Lastly, the organization of this thesis is as follows. Each chapter are pretty much independent and could be read in any order. Those who prefer the general picture first might start from Chapter 5 and then read Chapter 3, 4, and 6 in any order. Otherwise, one may also start from Chapter 3. Chapter 2 proves the results in Euclidean setting. Readers who would like to get a taste of the results may feel free to read it first. It can also be skipped without causing any problem at all. Chapter 3 reviews doubling spaces and doubling measures while Chapter 5 reviews Dirichlet spaces. Most of the results in these two chapters are from [6],[39],[45],[46],[47],[18] and [10]. Chapter 4 proves the doubling property for weighted measures and Chapter 6 gives the construction of weighted Dirichlet spaces and proves the heat kernel estimates. Those familiar with the subject can skip Chapter 3 and Chapter 5 altogether.

CHAPTER 2
A TASTE OF THE RESULTS

Before diving into the general theory of Dirichlet spaces, lets first take a look at the results on Euclidean spaces. All the results in this chapter will be proved again in the later chapter in the setting of metric spaces. So it can be skipped without causing any problems. The proofs though, will mostly be based upon calculus which make it interesting in itself.

In this chapter, $n \in \mathbb{N}$ is the fixed dimension of the Euclidean space $\mathbb{R}^n$. Consider the bilinear form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(\mathbb{R}^n, dx)$ defined by

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^n} \langle \nabla u, \nabla v \rangle dx = \int_{\mathbb{R}^n} \sum_{i=1}^{n} \partial_i u \cdot \partial_i v dx$$

with domain $\mathcal{D}(\mathcal{E}) = W^{1,2}(\mathbb{R}^n)$, the Sobolev space in $L^2(\mathbb{R}^n, dx)$.

Fixed a continuous function $a : [0,\infty] \to (0,\infty]$ satisfying

$$\sup_{[r.3r]} a \leq c_a \inf_{[r,3r]} a \quad \forall r > 0$$

for some constant c_a independent of r and define $h(x) = a(d(x,\Sigma))$ for any $x \in \mathbb{R}^n$. Here, Σ is an affine subspace of $\mathbb{R}^n$ with positive codimension, and d is the Euclidean distance on $\mathbb{R}^n$. By change of coordinate, we may assume that $\Sigma = \mathbb{R}^k \times \{0\}^{n-k}$ for some $k = 0, 1, ..., n-1$.

One example of such a is polynomial function with positive coefficients since the class of such a is closed under finite addition and multiplication. This class of functions is also closed under finite maximum and minimum. Therefore, one may construct new functions from old ones using these operations. Another note is that a satisfies the above inequality if and only if $1/a$ satisfy such inequality. Later on, it will be shown that such a are the quotient of two nondecreasing functions satisfying the above conditions.

Another example of a is $r \mapsto [\log(e + \frac{1}{r})]^{\alpha}$ for any fixed $\alpha > 0$. Since this function decrease slower than any polynomial, the function h associated to it is always locally integrable. The details proof of this is in the end of this chapter.

For any nonnegative function h, denote $\mathcal{E}^h(u, v) = \int_{\mathbb{R}^n} h\langle \nabla u, \nabla v\rangle dx$ for any $u, v \in C_c^{\infty}(\mathbb{R}^n)$. It will be shown in a later chapter that if h is locally integrable, then $(\mathcal{E}^h, C_c^{\infty}(\mathbb{R}^n))$ is closable and its closure, denoted by $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$, is a strongly local, regular Dirichlet space on $L^2(\mathbb{R}^n, hdx)$. Also denote d the Euclidean metric on $\mathbb{R}^n$.

The goal of this chapter is to give sufficient conditions for parabolic Harnack inequality of the weight Dirichlet space $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ defined above. This is equivalent to show that the weighted measure $d\mu = hdx$ is doubling and $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ satisfies (weak) Poincaré inequality for all balls[45]. Grigor'yan and Saloff-Coste characterize this further using classes of remote and anchored balls.

Fixed the singularity set Σ. A Σ-anchored ball is a ball centered in Σ, a Σ-remote ball is any ball $B(x, r)$ with $r \leq \frac{d(x,\Sigma)}{2}$. The prefix Σ is usually dropped if it can be understood from the context without causing any confusion.

A measure ν is said to satisfy the doubling condition for remote balls if for any remote ball $B(x, r)$, $\nu(B(x, r)) \leq C_D \nu(B(x, \frac{r}{2}))$ for some fixed constant $C_D \geq 1$, ν is said to satisfy doubling condition for anchored balls if for any anchored ball $B(x, r)$, $\nu(B(x, r)) \leq C_D \nu(B(x, \frac{r}{2}))$ for some fixed constant $C_D \geq 1$, ν is said to satisfy volume comparison condition if for any $r = d(x, \Sigma) = d(x, o) > 0, o \in \Sigma$,

$$\nu(B(o, r)) \leq C_V \nu(B(x, \frac{r}{64}))$$

for some fixed constant $C_V > 0$.

A Dirichlet form $(\mathcal{E}', \mathcal{D}(\mathcal{E}'))$ on $L^2(\mathbb{R}^n, \nu)$ with associated energy measure Γ is said to satisfy the (δ-weak) Poincaré inequality, $\delta \in (0, 1]$, for a family of balls $\mathcal{F}$ if there exists a fixed constant $C_P > 0$ such that for any $u \in \mathcal{D}_{loc}(\mathcal{E}')$, and any $B(x, r) \in \mathcal{F}$

$$\inf_{\xi \in \mathbb{R}} \int_{B(x,\delta r)} (u - \xi)^2 d\nu \leq C_P r^2 \int_{B(x,r)} d\Gamma'(u, u)$$

It is said to satisfy (weak) Poincaré inequality for remote(anchored) balls if $\mathcal{F}$ is the family of remote(anchored) balls.

The following two propositions are the main tools to prove the results in this chapter. For the prove of these two propositions, see [10, p.849,852]. Note that the setting in [10] is on weighted manifolds and the weight function is assumed to have no singularity. The proof, however, still work for weight functions with singularity. Actually, these two results hold for any regular, strongly local Dirichlet space(see Corollary 3.4.4 and Chapter 5).

Before proceeding, let the author summarises the assumptions for this chapter again. The weight function $h : \mathbb{R}^n \to (0, \infty]$ is a continuous function on $\mathbb{R}^n$ in the extended sense i.e. $h(x_k) \to h(x)$ whenever $x_k \to x$ no matter $h(x)$ is finite or not. The singularity set $\Sigma = \{h = \infty\}$ of h is assumed to have measure zero. This is to guarantee that the weighted Dirichlet form $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ is well-defined as a regular, strongly local Dirichlet form on $L^2(\mathbb{R}^n, hdx)$. Actually, $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ is the closure of $f, g \in C_c^\infty(\mathbb{R}^n) \mapsto \int_{\mathbb{R}^n} h\langle \nabla u, \nabla v\rangle dx$.

Proposition 2.0.2 *The weighted Dirichlet form $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ satisfies (weak) Poincaré inequality for all balls if and only if it satisfies (weak) Poincaré inequality for remote and anchored balls.*

Proposition 2.0.3 *A measure satisfies doubling condition for all balls if and only if it satisfies doubling condition for remote balls and volume comparison condition.*

The results of this chapter could be summarized as follows

Theorem 2.0.4 *Assume that the function $h : X \to (0, \infty]$ is given by $h(x) = a(d(x, \Sigma))$ where a satisfies*

$$\sup_{[r.3r]} a \leq c_a \inf_{[r,3r]} a \qquad \forall r > 0$$

and Σ is an affine subspace of dimension k. The weighted Dirichlet form $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ on $L^2(\mathbb{R}^n, d\mu = hd\nu)$ satisfies parabolic Harnack inequality if either one of the following conditions hold:

(*a*) *$k < n - 1$, and there exists a constant $c > 0$ such that $\int_0^r a(s)s^{n-k-1}ds \leq ca(r)r^{n-k}$ for any $r > 0$.*

(*b*) *$k = n-1$, and there exists a constant $c > 0$ such that $\int_s^r a(s)ds \leq c \min\{a(r), a(s)\}r$ for any $r > s > 0$.*

The proof of this theorem is separated into several parts. The first part is the proof of doubling property. The second is the proof of Poincaré inequality for several cases based on relationship between k and n. The case $k = 0$ and $n > 1$ gives the same result as that of Moschini and Tesei[27]. The proof, however, is mostly based on calculus.

As a consequence of this theorem, we have the following heat kernel estimates.

Corollary 2.0.5 *Under the assumption of the above theorem, the heat kernel $p(t, x, y)$ associated to $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ satisfies to following heat kernel estimates*

$$\frac{c_1}{A(t,x,y)t^{n/2}} e^{-|x-y|^2/c_2 t} \leq p(t,x,y) \leq \frac{c_3}{A(t,x,y)t^{n/2}} e^{-|x-y|^2/c_4 t}$$

where $A(t, x, y) = min\{a(\sqrt{t}), \sqrt{a(d(x,\Sigma)a(d(y,\Sigma)))}\}$ for some fixed constant $c_1, \ldots, c_4$.

The proof follows from the fact that $\mu(B(x,r)) \sim \min\{a(r), a(d(x,\Sigma))\}r^d$ uniformly in $r > 0$ and $x \in \mathbb{R}^n$.

2.1 Proof of the Doubling Property

Lemma 2.1.1 *If the weighted measure μ satisfy volume comparison condition, then it is doubling.*

Proof. It is sufficient to show that μ is doubling for remote balls. For any y in a remote balls $B(x,r)$, $|d(y,\Sigma) - d(x,\Sigma)| \leq r \leq \frac{d(x,\Sigma)}{2}$. Hence $d(y,\Sigma) \in [\frac{d(x,\Sigma)}{2}, \frac{3d(x,\Sigma)}{2}]$. It follows that

$$\sup_{B(x,r)} h \leq \sup_{[\frac{d(x,\Sigma)}{2}, \frac{3d(x,\Sigma)}{2}]} a \leq c_a \inf_{[\frac{d(x,\Sigma)}{2}, \frac{3d(x,\Sigma)}{2}]} a \leq c_a \inf_{B(x,r)} h$$

Therefore,

$$\mu(B(x,r)) \leq \sup_{B(x,r)} h \int_{B(x,r)} dx \leq 2^d c_a \inf_{B(x,r)} h \int_{B(x,\frac{r}{2})} dx \leq 2^d c_a \mu(B(x, \frac{r}{2}))$$

This proves the doubling condition for remote balls.

□

Theorem 2.1.2 *The weighted measure μ satisfies doubling condition for all balls if and only if there exists a constant $c > 0$ such that $\int_0^r a(s)s^{n-k-1}ds \leq ca(r)r^{n-k}$ for any $r > 0$.*

Proof. If the weighted measure μ satisfies doubling property, then it must also satisfies doubling comparison condition. On the contrary, volume comparison condition also implies doubling property by the previous lemma. Therefore, it is sufficient to show that the above condition is equivalent to volume comparison condition. This is obvious since for any $o \in \Sigma$, $\mu(B(o,r)) = \mu(B(0,r)) \sim r^k \int_0^r a(s)s^{n-k-1}ds$ while $\mu(B(x,r)) \sim a(d(x,\Sigma))r^n$ for any $r \leq d(x,\Sigma)/2$.

$\square$

2.2 Proof of (weak) Poincaré Inequality

Lets first prove the obvious result.

Lemma 2.2.1 *The weighted Dirichlet space $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ satisfies Poincaré inequality for remote balls.*

Proof. Let $B = B(x,r)$ be a remote ball and $u \in \mathcal{D}_{loc}(\mathcal{E}^h) \cap C_c(\mathbb{R}^n)$. Then $u \in \mathcal{D}(\mathcal{E})$ and

$$\begin{aligned}
\inf_{\xi\in\mathbb{R}} \int_B (u-\xi)^2 d\mu &\leq (\sup_{B(x,r)} h) \inf_{\xi\in\mathbb{R}} \int_B (u-\xi)^2 dx \\
&\leq c_a C_P r^2 (\inf_{B(x,r)} h) \int_B |\nabla u|^2 dx \\
&\leq c_a C_P r^2 \int_B |\nabla u|^2 d\mu
\end{aligned}$$

This finishes the proof.

$\square$

Next we prove Poincaré inequality for anchored balls which will immediately implies Poincaré inequality for all balls. The first result cover the case $n = 1$. It will be extended to the case $k = n - 1$ later on.

Theorem 2.2.2 *The weighted Dirichlet form $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ on $L^2(\mathbb{R}, \mu)$ satisfies Poincaré inequality for all balls whenever the following two conditions hold:*

(a) *μ satisfies doubling condition, i.e. whenever there exists a constant $c > 0$ such that $\int_0^r a(s)ds \le ca(r)r$ for any $r > 0$,*

(b) *a satisfies the following integral inequality: $\int_s^t a(r)dr \le ca(s)t$ for any $0 < s < t$, and some fixed constant $c > 0$.*

The condition (b) actually implies that a is equivalent to a decreasing function. To see this, fixed $t \ge 2s$. Then

$$\begin{aligned} \frac{t}{2}\sup_{[t,\frac{t}{2}]} a &\le \frac{c_a t}{2}\inf_{[t,\frac{t}{2}]} a \\ &\le c_a \int_s^t a(r)dr \\ &\le c_a c a(s)t \end{aligned}$$

This implies $\sup_{[t,\frac{t}{2}]} a \le 2c_a c a(s)$ for any $t \ge 2s$. Since this also hold for any $s < t < 2s$, one must have $a(t) \le 2c_a c a(s)$ for any $t > s$.

If one define $\tilde{a}(s) = \sup_{\{t>s\}} a(t)$, then $\frac{\tilde{a}}{2c_a c} \le a \le \tilde{a}$. It should be obvious that $\tilde{a}$ is decreasing.

Proof. Fixed $u \in C_c^\infty(\mathbb{R})$ and $r > 0$. Without loss of generality, one may assume $u(0) = 0$. Note that

$$\int_0^r u^2 d\mu = \int_0^r \Big(\int_0^x u'(y)dy\Big)^2 a(x)dx$$

$$
\begin{aligned}
&\leq \int_0^r \Big(x\int_0^x |u'|^2(y)dy\Big)a(x)dx \\
&= \int_0^r |u'|^2(y)\Big(\int_0^r xa(x)\chi_{\{y\leq x\}}dx\Big)dy \\
&\leq cr^2\int_0^r |u'|^2(y)a(y)dy
\end{aligned}
$$

By replacing u with $x \mapsto u(-x)$, one will also have $\int_{-r}^0 u^2 d\mu \leq cr^2\int_{-r}^0 |u'|^2(y)a(y)dy$. Therefore,

$$
\inf_{\xi\in\mathbb{R}}\int_{-r}^r (u-\xi)^2 d\mu = 2\int_{-r}^r u^2 d\mu \leq 2cr^2\int_{-r}^r |u'|^2 d\mu = cr^2\int_{-r}^r |u'|^2 d\mu
$$

□

The case $n \geq 2$ is based on polar coordinates. To prove it, one can not avoid proving Poincaré inequality for half line. So before moving on to higher dimension, lets prove the following result.

Lemma 2.2.3 *Let $(\mathcal{E}_N, \mathcal{D}(\mathcal{E}_N))$ be a Dirichlet form on $[0,\infty)$ with Neumann boundary condition and $\mathcal{E}_N(u) = \int_0^\infty |u'|^2 dx$ for any smooth function u. Let $b : [0,\infty) \to (0,\infty]$ be a locally integrable continuous function. Assume that b satisfies the following inequality:*

$$
\sup_{[r,3r]} b \leq c \inf_{[r,3r]} b,\ \forall r > 0
$$

and $\int_0^r b(s)ds \leq cb(r)r$, $\forall r > 0$, with a fixed constant $c > 0$. Then the weighted Dirichlet space $(\mathcal{E}_N^b, \mathcal{D}(\mathcal{E}_N^b))$ on $L^2([0,\infty), d\mu = bdx)$ satisfies Poincaré inequality for all balls.

The proof of this lemma is based on the following proposition.

Proposition 2.2.4 (e.g. [39], Lemma 5.3.12) *Fixed a doubling measure γ on a metric space (M,d). There exists a constant C_{DD}, depends only on doubling constant such that* *for any family of balls B_i in (M,d) and a sequence of nonnegative number b_i,*

$$\int \Big(\sum_i b_i \chi_{3B_i}\Big)^2 d\gamma \le C_{DD} \int \Big(\sum_i b_i \chi_{B_i}\Big)^2 d\gamma$$

Proof of Lemma 2.2.3. First notice that $d\mu = bdx$ is doubling, and $(\mathcal{E}_N^b, \mathcal{D}(\mathcal{E}_N^b))$ satisfies Poincaré inequality, with some fixed constant $P > 0$, on any interval $(x-s, x+s)$ with $s \le 3x/5$. Let $u \in \mathcal{D}(\mathcal{E}_N^h) \cap C_c([0,\infty))$. Set $\bar{u} = ⨍_0^r u d\mu$, $u_k = ⨍_{\frac{r}{2^{k+1}}}^{\frac{r}{2^{k-1}}} u d\mu$.

$$\begin{aligned}
\inf_{\xi \in \mathbb{R}} \int_0^r (u-\xi)^2 d\mu &\le \sum_{k=1}^{\infty} \int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^{k-1}}} (u-u_1)^2 d\mu \\
&\le 2\sum_{k=1}^{\infty} \int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^{k-1}}} (u-u_k)^2 d\mu + 2\sum_{k=1}^{\infty} \int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^{k-1}}} (u_k-u_1)^2 d\mu \\
&\le 2P\sum_{k=1}^{\infty} (\frac{3r}{2^{k+2}})^2 \int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^{k-1}}} |u'|^2 d\mu + 2\sum_{k=1}^{\infty} (u_k-u_1)^2 \int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^{k-1}}} d\mu \\
&\le Pr^2 \int_0^r |u'|^2 d\mu + 2\sum_{k=1}^{\infty} (u_k-u_1)^2 \mu([\frac{r}{2^{k+1}}, \frac{r}{2^{k-1}}])
\end{aligned}$$

Now,

$$\begin{aligned}
\mu([\frac{r}{2^k}, \frac{r}{2^{k+1}}])|u_k-u_{k+1}|^2 &= \int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}} |u_k-u_{k+1}|^2 d\mu \\
&\le 2\int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}} (u-u_k)^2 d\mu + 2\int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}} (u-u_{k+1}) \\
&\le 2P(\frac{3r}{8})^2 [\int_{\frac{r}{2^{k+1}}}^{\frac{r}{2^{k-1}}} |u'|^2 d\mu + \int_{\frac{r}{2^{k+2}}}^{\frac{r}{2^k}} |u'|^2 d\mu] \\
&\le Pr^2 \int_{\frac{r}{2^{k+2}}}^{\frac{r}{2^{k-1}}} |u'|^2 d\mu
\end{aligned}$$

and

$$\begin{aligned}
(u_k-u_1)^2 \chi_{[\frac{r}{2^{k+1}}, \frac{r}{2^{k-1}}]} &\le \Big(\sum_{j=1}^{k-1} |u_j - u_{j+1}| \chi_{[\frac{r}{2^{k+1}}, \frac{r}{2^{k-1}}]} \chi_{[0, \frac{3r}{2^{j+1}}]}\Big)^2 \\
&\le Pr^2 \Big[\sum_{j=1}^{k-1} \Big(\frac{1}{\mu([\frac{r}{2^j}, \frac{r}{2^{j+1}}])} \int_{\frac{r}{2^{j+2}}}^{\frac{r}{2^{j-1}}} |u'|^2 d\mu\Big)^{1/2} \chi_{[0, \frac{3r}{2^{j+1}}]}\Big]^2 \chi_{[\frac{r}{2^{k+1}}, \frac{r}{2^{k-1}}]}
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\sum_{k=1}^{\infty} \quad (u_k - u_1)^2 \quad & \mu([\frac{r}{2^{k+1}}, \frac{r}{2^{k-1}}]) \\
\leq \qquad & Pr^2 \int_0^r \Big[\sum_{j=1}^{\infty} \Big(\frac{1}{\mu([\frac{r}{2^j}, \frac{r}{2^{j+1}}])} \int_{\frac{r}{2^{j+2}}}^{\frac{r}{2^{j-1}}} |u'|^2 d\mu \Big)^{1/2} \chi_{[0, \frac{3r}{2^{j+1}}]} \Big]^2 \chi_{[\frac{r}{2^{k+1}}, \frac{r}{2^{k-1}}]} d\mu \\
\leq \qquad & 2Pr^2 \int_0^r \Big[\sum_{j=1}^{\infty} \Big(\frac{1}{\mu([\frac{r}{2^j}, \frac{r}{2^{j+1}}])} \int_{\frac{r}{2^{j+2}}}^{\frac{r}{2^{j-1}}} |u'|^2 d\mu \Big)^{1/2} \chi_{[0, \frac{3r}{2^{j+1}}]} \Big]^2 d\mu \\
\leq \qquad & 2PC_{DD} r^2 \int_0^r \Big[\sum_{j=1}^{\infty} \Big(\frac{1}{\mu([\frac{r}{2^j}, \frac{r}{2^{j+1}}])} \int_{\frac{r}{2^{j+2}}}^{\frac{r}{2^{j-1}}} |u'|^2 d\mu \Big)^{1/2} \chi_{[\frac{r}{2^{j+1}}, \frac{r}{2^j}]} \Big]^2 d\mu \\
= \qquad & 2PC_{DD} r^2 \int_0^r \sum_{j=1}^{\infty} \Big(\frac{1}{\mu([\frac{r}{2^j}, \frac{r}{2^{j+1}}])} \int_{\frac{r}{2^{j+2}}}^{\frac{r}{2^{j-1}}} |u'|^2 d\mu \Big) \chi_{[\frac{r}{2^{j+1}}, \frac{r}{2^j}]} d\mu \\
= \qquad & 2PC_{DD} r^2 \sum_{j=1}^{\infty} \int_{\frac{r}{2^{j+2}}}^{\frac{r}{2^{j-1}}} |u'|^2 d\mu \\
= \qquad & 2PC_{DD} r^2 \int_0^r |u'|^2 d\mu
\end{aligned}
$$

Combining all of these, we get

$$
\inf_{\xi \in \mathbb{R}} \int_0^r (u - \xi)^2 d\mu \leq (4C_{DD} + 1) Pr^2 \int_0^r |u'|^2 d\mu
$$

This prove Poincaré inequality for anchored balls, Hence we are done.

$\square$

Now assume $n \geq 2$ and $\Sigma = \{0\}$ a singleton set. One way to proof the results in this case is to follows Grigor'yan and Salof-Coste[10] arguments. See also [27]. The author, however, will decompose the space using polar coordinate. Even though this proof is specific to Euclidean space, it will also introduce a technique that will be useful in later chapter.

Theorem 2.2.5 *Assume $\Sigma = \{o\}$ and $h = a(d(\cdot, o))$ as before. The weighted Dirichlet form $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ on $L^2(\mathbb{R}^n, \mu)$ satisfies Poincaré inequality for all balls whenever*

μ satisfies doubling condition i.e. whenever there exists a constant $c > 0$ such that $\int_0^r a(s)s^{n-1}ds \le ca(r)r^n$ for any $r > 0$.

Proof. What's left is to prove Poincaré inequality for balls centered at 0. Fixed $r > 0$. Let $u \in C_c^\infty(\mathbb{R}^n)$ and use polar coordinate (s, θ). Denote $B = B(0, r)$, $u_s = u(s, \cdot)$, $\bar{u}_s = ⨍ u_s d\theta$, and $\bar{u} = ⨍_B u d\mu$.

$$\begin{aligned}
\inf_{\xi\in\mathbb{R}} \int_B (u-\xi)^2 d\mu &\le \int_B (u-\bar{u})^2 h dx \\
&\le 2[\int_B (u_s - \bar{u}_s)^2 a(s)s^{n-1} ds d\theta + \int_B (\bar{u} - \bar{u}_s)^2 a(s)s^{n-1} ds d\theta]
\end{aligned}$$

Using the Poincaré inequality on $\mathbb{S}^{n-1}$, one get $\int (u_s - \bar{u}_s)^2 n\theta \le C_1 \int |\partial_\theta u_s|^2 d\theta$. Therefore the first term is

$$\begin{aligned}
\int_B (u_s - \bar{u}_s)^2 a(s)s^{n-1} ds d\theta &\le C_1 \int_0^r \Big(\int |\partial_\theta u_s|^2 d\theta \Big) a(s)s^{n-1} d\theta \\
&\le C_1 \int_B |\partial_\theta u_s|^2 h dx \\
&\le 2C_1 r^2 \int_B |\nabla u|^2 d\mu
\end{aligned}$$

As for the second term, one applies Lemma 2.2.3 with $b(r) = a(r)r^{n-1}$ and get

$$\begin{aligned}
\int_0^r (\bar{u} - \bar{u}_s)^2 a(s)s^{n-1} ds &\le C_2 r^2 \int_0^r |\frac{d\bar{u}_s}{ds}|^2 a(s)s^{n-1} ds \\
&\le C_2 r^2 \int_0^r \Big(⨍ |\partial_s u| d\theta \Big)^2 a(s)s^{n-1} ds \\
&\le C_2 r^2 \int_0^r \Big(⨍ |\partial_s u|^2 d\theta \Big) a(s)s^{n-1} ds \\
&\le C_2 r^2 \int_B |\nabla u|^2 d\mu
\end{aligned}$$

□

Lastly, we gives the proof for the general case.

Theorem 2.2.6 *The weighted Dirichlet form $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ on $L^2(\mathbb{R}^n, \mu)$ satisfies parabolic Harnack inequality if either one of the following conditions hold:*

(a) *$k < n-1$, and there exists a constant $c > 0$ such that $\int_0^r a(s)s^{n-k-1}ds \leq ca(r)r^{n-k}$ for any $r > 0$.*

(b) *$k = n-1$, and there exists a constant $c > 0$ such that $\int_s^r a(s)s^{n-k-1}ds \leq c\min\{a(r), a(s)\}r^{n-k}$ for any $r > s > 0$.*

Again it can be proved in case (b) that the function a is equivalent to a decreasing function.

Proof. All one needs to do is to prove Poincaré inequality for anchored balls. The proof relies on $k = 0$ case.

Assume here that $k > 0$. By symmetry, it is sufficient to prove the result for balls centered at the origin. Fixed $r > 0$. From now on, we view $\mathbb{R}^n$ as product space $\mathbb{R}^k \times \mathbb{R}^{n-k}$, i.e. we identify $x = (y, z)$ for any $x \in \mathbb{R}^n$, $y \in \mathbb{R}^k$, and $z \in \mathbb{R}^{n-k}$. Moreover, $d\mu = a(|z|)dydz$.

Let $u \in C_c^\infty(\mathbb{R}^n)$. Denote $B = B^k(0, r) \times B^{d-k}(0, r)$, $u_z = u(\cdot, z)$, $\bar{u} = \fint_B u dx$, and $\bar{u}_z = \fint_{B^k(0,r)} u_z dy$. Here, $B^k(0, r)$ denote a ball in $\mathbb{R}^k$. Note that $\bar{u} = \fint_{B^{d-k}(0,r)} \bar{u}_z a(|z|)dz$, and hence

$$\begin{aligned}\int_{B^{d-k}(0,r)} (\bar{u}_z - \bar{u})^2 a(|z|)dz &\leq P_1 r^2 \int_{B^{d-k}(0,r)} \Big(\frac{d\bar{u}_z}{dz}\Big)^2 a(|z|)dz \\ &\leq P_1 r^2 \int_{B^{d-k}(0,r)} \fint_{B^k(0,r)} (\partial_z u)^2 a(|z|)dydz\end{aligned}$$

for some fixed constant P_1 independent of r and u. Also,

$$\int_{B^k(0,r)} (u_z - \bar{u}_z)^2 dy \leq P_2 r^2 \int_{B^k(0,r)} |\partial_y u|^2 dy$$

where P_2 is again independent of r and u. Therefore,

$$\begin{aligned}
\inf_{\xi\in\mathbb{R}} \int_B (u-\xi)^2 d\mu &\leq \int_B (u-\bar{u})^2 d\mu \\
&\leq 2\int_{B^{d-k}(0,r)} a(|z|) \int_{B^k(0,r)} (u_z - \bar{u}_z)^2 dydz \\
&\quad +2\int_{B^k(0,r)} \int_{B^{d-k}(0,r)} (\bar{u}-\bar{u}_z)^2 a(|z|) dzdy \\
&\leq 2P_2 r^2 \int_{B^{d-k}(0,r)} a(|z|) \int_{B^k(0,r)} |\partial_y u|^2 dydz \\
&\quad +2P_1 r^2 \int_{B^{d-k}(0,r)} \fint_{B^k(0,r)} (\partial_z u)^2 a(|z|) dydz \int_{B^k(0,r)} dz \\
&\leq 2(P_1+P_2) r^2 \int_B |\nabla u|^2 d\mu
\end{aligned}$$

Now, $B^k(0,\frac{r}{2}) \times B^{n-k}(0,\frac{r}{2}) \subset B^n(0,r) \subset B^k(0,r) \times B^{n-k}(0,r)$. Hence,

$$\inf_{\xi\in\mathbb{R}} \int_{B(0,r)} (u-\xi)^2 d\mu \leq 2(P_1+P_2) r^2 \int_{B(0,2r)} |\nabla u|^2 d\mu$$

□

Corollary 2.2.7 *Let Σ be an affine subspace of $\mathbb{R}^n$ with dimension k and $h(x) = d(x,\Sigma)^\alpha$ with $\alpha < 0$. Then the weighted Dirichlet Space $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ satisfies parabolic Harnack inequality if and only if $\alpha > n-k$.*

Proof. Under the assumption $\alpha < 0$, the weighted measure $d\mu = hdx$ is doubling if and only if $\alpha > n-k$.

□

Lastly, lets end this chapter with an example of a weight function that always work on any Euclidean spaces regardless of its dimension.

2.3 Example

In this section, let $\alpha > 0$ be fixed and denote $a(r) = a_\alpha(r) = [\log(e + \frac{1}{r})]^\alpha$ for any $r > 0$. Denote $h = a(d(\cdot, \Sigma))$ where Σ is a k-dimension affine subspace of $\mathbb{R}^n$. The goal is to show that

Theorem 2.3.1 *Given*

$$\mathcal{E}^\alpha(u, v) = \int_{\mathbb{R}^n} a(d(x, \Sigma))\langle \nabla u, \nabla v\rangle(x)dx = \int_{\mathbb{R}^n} \sum_{i=1}^{n} a(d(x, \Sigma))\partial_i u(x) \cdot \partial_i v(x)dx$$

with domain $\mathcal{D}(\mathcal{E}^\alpha) = W^{1,2}(\mathbb{R}^n, a(d(\cdot, \Sigma))dx)$*, the weighted Sobolev space. Then* $(\mathcal{E}^\alpha, \mathcal{D}(\mathcal{E}^\alpha))$ *satisfies parabolic Harnack inequality for all* $\alpha > 0$ *and* $n = 1, 2, \ldots$ *Moreover, its heat kernel* $p(t, x, y)$ *satisfies the following estimates*

$$\frac{c_1}{A(t, x, y)t^{n/2}} e^{-|x-y|^2/c_2 t} \leq p(t, x, y) \leq \frac{c_3}{A(t, x, y)t^{n/2}} e^{-|x-y|^2/c_4 t}$$

where $A(t, x, y) = \left(min\{-\log(t\wedge 1/2),\ \sqrt{\log(d(x, \Sigma)\wedge 1/2)\log(d(y, \Sigma)\wedge 1/2)}\}\right)^{\alpha/2}$ *for some fixed constant* $c_1, \ldots, c_4$*.*

This can be done by showing that there exists a constant $c > 0$ such that $\int_s^r a(s)s^{n-k-1}ds \leq c\min\{a(r), a(s)\}r^{n-k}$ for any $r > s > 0$. Since a is decreasing, we obviously have $\int_s^r a(s)s^{n-k-1}ds \leq a(s)r^{n-k}$ for any $r > s > 0$. Therefore it is sufficient to prove there exists a constant $c > 0$ such that $\int_s^r a(s)s^{n-k-1}ds \leq ca(r)r^{n-k}$ for any $r > s > 0$.

Lemma 2.3.2 *The logarithmic weighted function a defined earlier satisfies*

$$\sup_{[r,3r]} a \leq c_a \inf_{[r,3r]} a, \forall r > 0$$

for some constant $c_a < 3^\alpha$

Proof. Note that $a' < 0$, so $\sup_{[r,3r]} a = a(r)$ and $\inf_{[r,3r]} a = a(3r)$. Therefore, it is sufficient to show that $c_a = \sup_{r>0} \frac{a(r)}{a(3r)} < 3^\alpha$. Without loss of generality, one may assume $\alpha = 1$. Denote

$$A(r) = \frac{a(r)}{a(3r)} = \frac{\log(e + \frac{1}{r})}{\log(e + \frac{1}{3r})}$$

Then $\lim_{r\to\infty} A(r) = 1$. Also

$$\lim_{r\to 0} A(r) = \lim_{r\to 0} \frac{-1/(r^2(e + \frac{1}{r}))}{-1/(3r^2(e + \frac{1}{3r}))} = \lim_{r\to 0} \frac{3er + 1}{er + 1} = 1$$

Since $A > 1$, A must attains a finite maximum. Next assume that A attains maximum at $R \in (0, \infty)$. Using the fact that

$$A'(r) = \frac{\frac{-\log(e+\frac{1}{3r})}{(er^2+r)} + \frac{\log(e+\frac{1}{r})}{(3er^2+r)}}{[\log(e + \frac{1}{3r})]^2}$$

and $A'(R) = 0$, one must have $A(R) = \frac{3eR^2+R}{eR^2+R} < 3$.

□

The above proof also gives the following result.

Corollary 2.3.3 *Fixed $c > 0$. Define $A(r) = \frac{\log(e+\frac{1}{r})}{\log(e+\frac{1}{cr})}$ for $r > 0$. Then $\sup_r A(r) < \infty$. Moreover, $\sup_r A(r) < c$ if $c > 1$.*

Now lets proof the integral Inequality: $\int_0^r a(s)ds \le ca(r)r$. This will implies a more general inequality $\int_0^r a(s)s^k ds \le ca(r)r^{k+1}$.

For $\alpha \le 1$, there is a simple proof of this inequality.

Theorem 2.3.4 *For $\alpha \le 1$, there exists a constant $C \in (0, \frac{2c_a}{3-c_a}]$ such that $\int_0^r a(s)ds \le Ca(r)r$.*

The sole reason for a requirement $\alpha \le 1$ is to guarantee that $c_a < 3$.

Proof. Fixed $r > 0$,

$$\begin{aligned}
\int_0^r a(s)ds &= \sum_{k=1}^{\infty} \int_{\frac{r}{3^k}}^{\frac{r}{3^{k-1}}} a(s)ds \\
&\le a(r) \sum_{k=1}^{\infty} c_a^k \int_{\frac{r}{3^k}}^{\frac{r}{3^{k-1}}} ds \\
&= 2a(r)r \sum_{k=1}^{\infty} \frac{c_a^k}{3^k} \\
&= 2\frac{c_a}{3 - c_a} a(r)r
\end{aligned}$$

□

For general $\alpha > 0$, the proof is more complicated.

Theorem 2.3.5 *For any $r > 0$, $\int_0^r a(s)ds \le (e^\alpha \vee 2)a(r)r$.*

Proof. First, one shows that there exists $C > 0$ such that for any $r > 0$, $\int_0^r a(s)ds \le Ca(r)r$. Fixed $k > 0$ small enough so that $k\alpha < 1$. For any $r < \frac{1}{e} \wedge \frac{1}{e^\alpha}$,

$$\begin{aligned}
\int_0^r a(s)ds &\le \frac{1}{k^\alpha} \int_0^r [\log(\frac{2}{s})^k]^\alpha ds \\
&\le \frac{1}{k^\alpha} \int_0^r \frac{2^{k\alpha}}{s^{k\alpha}} ds \\
&\le \frac{2^{k\alpha}}{k^\alpha} r^{1-k\alpha}
\end{aligned}$$

Choose $k = -\frac{1}{\log r}$, then

$$\inf_{0<r\le\frac{1}{e}} kr^k \log(e + \frac{1}{r}) = \inf_{1\le x<\infty} \frac{1}{x}(e^{-x})^{1/x} \log(e + e^x) = e^{-1}$$

Hence, $\int_0^r a(s)ds \le 2^{-\alpha/\log r} e^\alpha a(r)r$. For $r \ge \frac{1}{e} \wedge \frac{1}{e^\alpha}$, use the fact that $1 \le a(r) \le a(\frac{1}{e} \wedge \frac{1}{e^\alpha})$ and conclude

$$\int_0^{\frac{1}{e}\wedge\frac{1}{e^\alpha}} a(s)ds + \int_{\frac{1}{e}\wedge\frac{1}{e^\alpha}}^r a(s)ds \le 2e^\alpha a(\frac{1}{e} \wedge \frac{1}{e^\alpha})a(r)r + a(\frac{1}{e} \wedge \frac{1}{e^\alpha})a(r)r$$

Next, one finds the best C possible. Set $A(r) = \frac{1}{a(r)r}\int_0^r a(s)ds$. By the previous conclusion $C = \sup_{r>0} A(r) < \infty$.

Case I: A does not attain the maximum. Then either there exists a sequence $r_k \to 0$ such that $A(r_k) \to C$ or there exists a sequence $s_k \to \infty$ such that $A(s_k) \to C$. In the first case,$\lim_{k\to\infty} A(r_k) \leq \lim_{k\to\infty} 2^{-\alpha/\log r_k} e^{\alpha} = e^{\alpha}$. In the latter case, fixed r_0 so that $a(s) < 3^{\alpha}/2$ for all $s > r_0$. We have for any $r > 2Cr_0$,

$$\int_0^r a(s)ds \leq Ca(r_0)r_0 + a(r_0)(r - r_0) \leq 3^{\alpha} r \leq 3^{\alpha} ra(r)$$

Therefore, $\lim_{k\to\infty} A(s_k) \leq 3^{\alpha}$.

Case II: A attains a maximum at $R \in (0, \infty)$. Then $A'(R) = 0$. But

$$A'(r) = \frac{a^2(r)r - (a'(r)r + a(r))\int_0^r a(s)ds}{a^2(r)r^2}$$

Therefore, $A(R) = \frac{a(R)}{a'(R)R + a(R)}$. Note that

$$a'(r) = \frac{[\log(e + \frac{1}{r})]^{\alpha-1}}{(-r^2(e + \frac{1}{r}))} = -\frac{a(r)}{(er^2 + r)\log(e + \frac{1}{r})}$$

Therefore

$$\begin{aligned} A(R) &= \frac{a(R)}{a(R) - \frac{a(R)}{(eR+1)\log(e+\frac{1}{R})}} \\ &= \frac{(eR+1)\log(e + \frac{1}{R})}{(eR+1)\log(e + \frac{1}{R}) - 1} \\ &= 1 + \frac{1}{(eR+1)\log(e + \frac{1}{R}) - 1} \end{aligned}$$

Set $B(r) = (er + 1)\log(e + \frac{1}{r}) - 1$. Since $\lim_{r\to 0,\infty} B(r) = \infty$, B attains a minimum at some point $r_0 \in (0, \infty)$. Now

$$0 = B'(r_0) = (er_0 + 1)/(-r_0^2(e + \frac{1}{r_0})) + e\log(e + \frac{1}{r_0})$$

which implies $\log(e + \frac{1}{r_0}) = \frac{1}{er_0}$. So $B(r_0) = 1 + \log(e + \frac{1}{r_0}) - 1 \geq 1$. This concludes that $C = A(R) \leq 2$.

□

This proves Theorem 2.3.1 and ends this chapter. The next chapter reviews background results on doubling spaces and doubling measures. Readers familiar with the subjects may skip to Chapter 4 right away.

CHAPTER 3

DOUBLING SPACES

This chapter introduces the doubling property, one of the two neccessary conditions for both parabolic Harnack inequality and heat kernel estimates. Another condition is the Poincaré inequality which will be covered in Chapter 5. It turns out that doubling property and Poincaré inequality together are sufficient to prove heat kernel estimates and parabolic Harnack inequality[47].

Unlike the Poincaré inequality, the doubling property does not depend directly on the Dirichlet form. Rather, it depends on the geometry of the space. This allow us to discuss the doubling property without the need to discuss Dirichlet spaces.

Most results of this chapter are labored from [19],[8],[18], and [10], or are direct consequence of the results contained in these references. There are two versions of doubling property, one for spaces, and another for measures.

3.1 Doubling Spaces

Definition 3.1.1 (Doubling spaces) *A metric space (X, d) is doubling if there exists a constant $N_X \in \mathbb{N}$ such that any ball of radius r can be covered by N_X balls of radius $\frac{r}{2}$.*

Note that doubling property is a property of finite dimensional spaces, open subsets of infinite dimensional spaces cannot be doubling.

Recall that a subset of a metric space is totally bounded if it can be covered by a finite number of balls with arbitrarily small fixed radius.

Proposition 3.1.2 *Let (X, d) be a doubling space.*

(a) *A subset of X is bounded if and only if it is totally bounded.*

(b) *If (X, d) is complete, any ball in X is compact. Hence, (X, d) is locally compact.*

Proof. To prove (a), it is sufficient to show that all ball in X is totally bounded. This follows easily by applying the assumption successively: any ball $B(x, r)$ in X can be covered by N_X^k balls of radius $\frac{r}{2^k}$.

To prove (b), recall that a metric space is compact if and only if it is complete and totally bounded. Since each ball $B(x, r)$ is complete, the result follows from (a)

□

Next we characterize the doubling spaces.

Theorem 3.1.3 *Let (X, d) be a metric space. The following are equivalent.*

(a) *(X, d) is doubling.*

(b) *There is a function $N_X : (0, 1/2] \to (0, \infty)$ such that any ball of radius r can be covered by $\lfloor N_X(\varepsilon) \rfloor$ balls of radius εr.*

(c) *There is a $\alpha > 0$, $c = c_\alpha \geq 1$ such that for any fixed $\varepsilon \in (0, 1/2]$, any ball of radius r can be covered by $\lfloor c\varepsilon^{-\alpha} \rfloor$ balls of radius εr.*

Proof. Clearly, (c) $\implies$ (b) $\implies$ (a) To prove (a) $\implies$ (c), set $\alpha = \log_2 N_X$ and $c = N_X$. For each $0 < \varepsilon \leq 1/2$, choose $k \in \mathbb{N}$ so that $\frac{1}{2^{k+1}} \leq \varepsilon < \frac{1}{2^k}$. Easy calculation show that $N_X^{k+1} = c2^{k\alpha} \leq c\varepsilon^{-\alpha}$. Iterating the assumption, any ball of radius r can be covered by N_X^{k+1} balls of radius $\frac{r}{2^{k+1}} \leq \varepsilon r$ and the result follows.

$\square$

Definition 3.1.4 (Assouad Dimension) *Let (X, d) be a doubling space. The Assouad dimension of X, denotes $\dim_A X$, is the finite infimum of $\alpha > 0$ so that there exists $c_\alpha \geq 1$ with the following property: any ball of radius r can be covered by $\lfloor c_\alpha \varepsilon^{-\alpha} \rfloor$ balls of radius εr.*

Note that $\dim_A X \leq \log_2 N_X$.

Proposition 3.1.5 *The completion of a doubling space is doubling and also has the same Assouad dimension.*

Proof. Obvious.

$\square$

Proposition 3.1.6 *Let (X, d) be a doubling space. The Assouad dimension of X is equal to the infimum of all $\beta > 0$ such that there exists a corresponding $c_\beta \geq 1$ with the following property: for any fixed $0 < \varepsilon \leq 1/2$, any ball of radius r has at most $\lfloor c_\beta \varepsilon^{-\beta} \rfloor$ disjoint points of mutual distance at least εr.*

Proof. Let β be as above and choose $\{x_i\}$ a subset of $B(x, r)$ so that $d(x_i, x_j) \geq \varepsilon r$ for all $i \neq j$. WLOG, one may assume $\{x_i\}$ is maximized in the sense that no point can be added to $\{x_i\}$ so that $\min_{i \neq j} d(x_i, x_j)$ remains at least εr. Then $B(x_i, \varepsilon r)$ must be a covering of $B(x, r)$. Hence $\dim_A X \leq \beta$.

To prove the converse, fixed $\beta > \dim_A X$ and let S be a maximal subset of $B(x,r)$ such that $d(s, S-\{s\}) \geq \varepsilon r$. Choose an open covering $B_i = B(x_i, \frac{\varepsilon}{2}r)$, $i = 1,\ldots,n$ with $n \leq \lfloor c_\beta 2^\beta \varepsilon^{-\beta} \rfloor$. Note that each B_i can only contain at most one $s \in S$. Hence S must have at most $\lfloor c_\beta 2^\beta \varepsilon^{-\beta} \rfloor$ elements.

□

Let's finish this section with the existence of homogeneous measures for doubling spaces. Here an α-homogeneous measure is a Borel measure μ such that there exists a constant $c = c_\mu \geq 1$ with

$$\frac{\mu(B(x,r))}{\mu(B(x,s))} \leq c\left(\frac{r}{s}\right)^\alpha$$

for any $x \in X$ and $0 < s < r$.

Proposition 3.1.7 (e.g. [18]) *Any complete doubling space carries an α-homogeneous measure for each α larger that the Assouad dimension.*

3.2 Doubling Geodesic Spaces

Let's start this section with the definition of length of a path between two points.

Definition 3.2.1 *Let $\gamma : [0,1] \to (X,d)$ be a path in a metric space. The length of γ is defined to be the supremum of*

$$\sum_{i=1}^{k} d(\gamma(t_{i-1}), \gamma(t_i))$$

where the supremum is taken over all partitions $0 = t_0 < t_1 < \ldots < t_k = 1$.

A metric space is said to be a length space if the distance between any two points is the infimum of the length of all paths connecting those two points.

A complete length space is a geodesic space if one can always find a path connecting any two points with the length equal to their distance.

Proposition 3.2.2 *Any complete doubling length space is a geodesic space.*

Proof. This follows from the fact that any complete doubling space is locally compact(Proposition 3.1.2).

□

One of nice features of doubling geodesic spaces is the existance of Whitney covering on any open sets.

Definition 3.2.3 *Fixed an $\epsilon > 0$. A strict ϵ-Whitney covering of an open subset U in a metric space (X, d) is any family $\mathcal{W}$ of disjoint balls such that*

(a) $\cup_{B\in\mathcal{W}} 3B = U$ *where* $kB(x, r) = B(x, kr)$ *for any* $k, r \geq 0$ *and* $x \in X$*,*

(b) *for any* $B = B(x, r) \in \mathcal{W}$, $r = \epsilon d(B, X - U)$*,*

Proposition 3.2.4 (e.g. [39], [17],[25]) *If (X, d) is a doubling geodesic space, then ones can always construct a strict ϵ-Whitney covering $\mathcal{W}$ for an open subset U whenever $\epsilon < 1/4$. Moreover, $\mathcal{W}$ satisfies the following extra properties*

(a) *the family $\mathcal{W}$ is countable,*

(b) *there is a finite constant $a = a_\epsilon$ such that for any $k \leq \frac{1}{10\epsilon}$, $\sum_{B\in\mathcal{W}} \chi_{kB} \leq a$.*

3.3 Doubling Measures

In this section, the author assumes that any topological space is path-connected, Hausdorff, locally compact, and second countable, hence metrizable. The author also uses the term $LCHS$ space to referred to such spaces. A ball centered at x and of radius r will be denote by $B(x,r)$. Note that r is always chosen so that $B(x,s) \neq B(x,r)$ for all $s < r$.

Recall that a σ-field or a σ-algebra $\mathcal{A}$ on a LCHS space X is a collection of subsets that is closed under countable unions, complement and contain the whole space X. A Borel σ-field $\mathcal{B}(X)$ is the smallest σ-field containing all open sets of X. An element of $\mathcal{B}(X)$ is called a Borel set. A Borel σ-field is always exists and is closed under countable intersection as well.

A Borel measure on a LCSH space X is a function $\mu : \mathcal{B}(X) \to [0,\infty]$ such that for any disjoint Borel sets $A_i \in \mathcal{B}(X)$, $\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

A support of a measure is the smallest close set S such that $\mu(X - S) = 0$. A measure is said to have full support if its support is X.

A radon measure is a Borel measure such that $\mu(K) < \infty$ for any compact subset K of X.

Definition 3.3.1 *Denote $\mathcal{F}$ a family of balls in a metric space X. A Borel measure μ on a metric space (X,d) is said to satisfy volume doubling property, or doubling property for short, on $\mathcal{F}$ if there exists a constant $C_D \geq 1$ such that*

$$\mu(B(x,r)) \leq C_D \mu(B(x,r/2)),\ \forall B(x,r) \in \mathcal{F}$$

Sometimes we write $\mu \in (VD)$ if the measure μ satisfies doubling property for all *balls.*

Well-known examples of doubling measures are Lebesgue measures on Euclidean spaces, Haar measures on virtually nilpotent Lie groups, Riemannian volumes of Riemannian manifolds with nonnegative Ricci curvature. See, for example, [41] and [31].

Proposition 3.3.2 *For any $\mu \in (VD)$, either μ is zero or it is a σ-finite radon measure with full support.*

Proof. Assume that $\mu(B(x,r)) = 0$ for some $x \in X$, $r > 0$. By doubling property, $\mu(B(x,2^k r)) \leq C_D^k \mu(B(x,r)) = 0$ for all k. Taking $k \to \infty$, we have $\mu(X) = 0$. Thus, either we have $\mu = 0$ or μ has full support. Assume that $\mu \neq 0$. If $\mu(B(x,r)) = \infty$, then $\mu(B(x,2^{-k}r)) \geq C_D^{-k}\mu(B(x,r)) = \infty$ for all k and hence $\lim_{k\to\infty} \mu(B(x,2^{-k}r)) = \infty$, contradicts to the continuity of measure. Therefore $\mu(B(x,r)) < \infty$ for any ball $B(x,r)$. Particularly, μ is radon. Since X is second countable, we also have μ is σ-finite.

□

Obviously, homogeneous measure is doubling. It turn out that the converse is also true. Moreover, if the underlying metric is geodesic, then doubling measures must at least grow polynomially.

Proposition 3.3.3 ([39]) *Fixed $\mu \in (VD)$ with doubling constant C_D and denote $\alpha = \log_2 C_D$. Then for any $s < r$, $x, y \in X$, we have*

$$\frac{\mu(B(x,r))}{\mu(B(y,s))} \leq C_D\Big(\frac{r + d(x,y)}{s}\Big)^{\alpha}$$

Proof. First we assume $x = y$, choose $k \in \mathbb{Z} \cup \{0\}$ so that $2^k \le r/s \le 2^{k+1}$. Then $C_D^k = 2^{k\alpha} \le (r/s)^\alpha$ and

$$\begin{aligned}
\mu(B(x,r)) &\le \mu(B(x, 2^{k+1}s)) \\
&\le C_D^{k+1}\mu(B(x,s)) \\
&\le C_D\Big(\frac{r}{s}\Big)^\alpha \mu(B(x,s))
\end{aligned}$$

For general case, we use the fact that $B(x,r) \subset B(y, r + d(x,y))$ to conclude that $\mu(B(x,r)) \le C_D[(r + d(x,y))/s]^\alpha \mu(B(y,s))$.

□

Proposition 3.3.4 ([39]) *Let μ be a Borel measure in a geodesic space. Assume that $\mu \in (VD)$ with doubling constant C_D. Denote $\beta = \log_3(1 + C_D^{-3})$, and $c_D = (1 + C_D^{-3})^{-1}$. Then for any $s < r$, $x \in X$ with $B(x,r) \neq M$, we have*

$$\frac{\mu(B(x,r))}{\mu(B(x,s))} \ge c_D\Big(\frac{r}{s}\Big)^\beta$$

Proof. Pick $z \in M - B(x,r)$ and choose a path γ from x to z. Since the function $t \mapsto d(x, \gamma(t))$ is continuous, there exists $y = \gamma(t_0)$ such that $d(x,y) = 2r/3$. This implies $B(y, r/3)$ and $B(x, r/3)$ are disjoint. Moreover, $\mu(B(x, r/3)) \le 3^{\log_2 C_D} C_D \mu(B(y, r/3)) \le C_D^3 \mu(B(y, r/3))$. Therefore

$$\mu(B(x,r)) \ge \mu(B(x, r/3)) + \mu(B(y, r/3)) \ge (1 + C_D^{-3})\mu(B(x, r/3))$$

For general $s < r$, choose $k \in \mathbb{N}$ so that $3^k \le r/s \le 3^{k+1}$. Then

$$\begin{aligned}
\mu(B(x,r)) &\ge (1 + C_D^{-3})^k \mu(B(x, r/3^k)) \\
&\ge \frac{3^{(k+1)\log_3(1+C_D^{-3})}}{1 + C_D^{-3}} \mu(B(x,s)) \\
&\ge (1 + C_D^{-3})^{-1}\Big(\frac{r}{s}\Big)^{\log_3(1+C_D^{-3})} \mu(B(x,s))
\end{aligned}$$

$\square$

The following result is also from [39]. The proof is based on Marcinkiewicz interpolation theorem. It will be used in the proof of Poincaré inequality later on.

Proposition 3.3.5 (e.g. [39] Lemma 5.3.12) *Assume that μ is a doubling measure in a geodesic space. Then for any $K > 0$, there exists a constant $C > 0$ such that for any sequence B_i of balls and nonnegative numbers a_i,*

$$\int \Big(\sum_i a_i \chi_{KB_i} \Big)^2 \leq C \int \Big(\sum_i a_i \chi_{B_i} \Big)^2$$

The next result show that doubling measures only exists in doubling spaces.

Corollary 3.3.6 *Fixed $\mu \in (VD)$ and $r > 0$. For any $\delta > 0$, there is a number $K_\delta \in \mathbb{N}$ such that for any relatively compact ball of radius r can be covered by at most K_δ balls of radius δr.*

Proof. Fixed a relatively compact ball $B = B(x, r)$ and $x_0 \in B$. For any $k > 0$, choose $x_k \in B - \cup_{i=0}^{k} B(x_i, s)$. Since B is relatively compact, this process must stop, say at K. Clearly, $B \subset \cup_{i=0}^{K} B(x_i, s)$. Choose k so that $\mu(B(x_k, s)) = \min_{0 \leq i \leq K} \mu(B(x_i, s))$. Using doubling property and the fact that $B(x_i, s/2)$ are disjoint,

$$\begin{aligned} K &\leq \frac{1}{\mu(B(x_k, s/2))} \sum_{i=0}^{K} \mu(B(x_i, s/2)) \\ &\leq \frac{\mu(B(x, r + s/2))}{\mu(B(x_k, s/2))} \\ &\leq C_D \Big(\frac{r + s/2 + d(x, x_k)}{s/2}\Big)^{\alpha} \\ &\leq C_D \Big(\frac{4 + \delta}{\delta}\Big)^{\alpha} \end{aligned}$$

Thus, we can set $K_\delta = \lfloor C_D(\frac{4+\delta}{\delta})^\alpha \rfloor$.

$\square$

3.4 Doubling Property: From remote balls to all balls

The original idea of results in this and the next section belongs to A. Grigor'yan and L. Saloff-coste [10].

Definition 3.4.1 *Fixed $\varepsilon, \lambda \in (0, 1]$, and Σ a closed subset in X. A Borel measure μ on X is said to satisfy volume comparison property on Σ with parameter (ε, Σ), or $\mu \in (VC)_{\Sigma,\varepsilon,\lambda}$ for short, if there is a constant $C_V > 0$ such that for any $o \in \Sigma$ and any $x \in X - \Sigma$ such that $d(x, \Sigma) \geq \lambda d(o, x)$,*

$$\mu(B(o, d(o, x))) \leq C_V \mu(B(x, \frac{1}{32}\varepsilon d(o, x)))$$

Note that in many cases, for example when μ has doubling property for remote balls, the constant $1/32$ is not particularly important.

Lemma 3.4.2 *If $\mu \in (VD)$, then $\mu \in (VC)_{\Sigma,\varepsilon,\lambda}$ for any closed set Σ and parameter λ, ε.*

Proof.

$$\begin{aligned}
\frac{\mu(B(o, d(x, o)))}{\mu(B(x, \frac{1}{32}\varepsilon d(o, x)))} &\leq C_D\Big(\frac{d(x, o) + d(x, o)}{\frac{1}{32}\varepsilon d(x, o)}\Big)^\alpha \\
&\leq C_D(64\varepsilon^{-1})^\alpha \\
&= 64^\alpha C_D \varepsilon^{-\alpha}
\end{aligned}$$

□

Recall that a Ball $B(x,r)$ is said to be ε-remote to Σ if $r \le \frac{1}{2}\varepsilon d(x,\Sigma)$ and is said to be Σ-anchored balls if its center lies in Σ.

Theorem 3.4.3 *Assume that μ satisfies doubling property for ε-remote balls. Then $\mu \in (VD)$ if and only if μ satisfies doubling property for Σ-anchored balls and $\mu \in (VC)_{\Sigma,\varepsilon,\lambda}$ for some $\lambda \in (0,1]$.*

Proof. We only need to prove sufficiency part. Let $B(x,r)$ be a non anchored ball and $\rho = d(x,\Sigma)$. Choose $o \in \Sigma$ so that $d(x,o) = d(x,\Sigma)$. If $r \le \varepsilon\rho/2$, then $B(x,r)$ is a remote ball. If $r > 3\rho$, then $B(x,r) \subset B(o,\frac{4}{3}r)$ and $B(o,\frac{1}{6}r) \subset B(x,\frac{1}{2}r)$. Using doubling property for anchored balls, we have

$$\mu(B(x,r)) \le C_D^3\mu(B(o,\frac{1}{6}r)) \le C_D^3\mu(B(x,\frac{1}{2}r))$$

If $\frac{1}{2}\varepsilon\rho \le r \le 3\rho$, $B(x,r) \subset B(o,4\rho)$ and $B(x,\frac{1}{32}\varepsilon\rho) \subset B(x,\frac{1}{2}r)$. By (VD) and $(\mathrm{VC})_{\Sigma,\varepsilon,\lambda}$,

$$\mu(B(x,r)) \le C_D^2 C_V \mu(B(x,\frac{1}{2}r))$$

□

The last result in this chapter is from [10]. Saloff-Coste and Grigor'yan characterize the doubling property using volume comparison on a fully accessible set. This result will be generalized in the next chapter. Recall that a fully accessible set is a closed set Σ such that for any $o \in \Sigma$ and $r > 0$, there is $x \in X$ with $d(x,\Sigma) = d(o,x) = r$. An example of a fully accessible set is a singleton, any vector subspace of $\mathbb{R}^n$ of positive codimension. Another example is $\Sigma = \partial V$ in $X = \mathbb{R}^n - V$ where V is an open convex subset of $\mathbb{R}^n$.

Corollary 3.4.4 ([10]) *Let μ be a Borel measure on a geodesic space (X, d). Assume that μ satisfies doubling property for balls ε-remoted to a fully accessible subset Σ. Then $\mu \in (VD)$ if and only if $\mu \in (VC)_{\Sigma,\varepsilon,\lambda}$.*

Proof. It is sufficient to show doubling property for anchored balls. Let $o \in \Sigma$ and $r > 0$. Since Σ is fully accessible, there is $x \in X$ with $d(x, \Sigma) = d(x, o) = r$. Choose a path γ from x to o with length at most $(1 + \delta)r$. Choose $0 = t_0 \leq \ldots \leq t_i \leq t_{i+1} \leq \ldots \leq t_k$ so that $d(\gamma(t_i), \gamma(t_{i+1})) = \frac{1}{32}\varepsilon r$ and $d(o, \gamma(t_k)) = \min_i d(o, \gamma(t_i)) = r/4$. Clearly, $k \leq 32[3/4 + \delta]\varepsilon^{-1}$. It is not hard to see that each ball $B(\gamma(t_i), \epsilon r/8)$ is ε-remote to Σ. Since $B(\gamma(t_i), \varepsilon r/32) \subset B(\gamma(t_i), \varepsilon r/8)$, $\mu(B(\gamma(t_i), \varepsilon r/32)) \leq C_D^2 \mu(B(\gamma(t_{i+1}), \varepsilon r/32))$. Therefore,

$$\begin{aligned}
\mu(B(o, r)) &\leq C_V \mu(B(x, \varepsilon r/32)) \\
&\leq C_V C_D^k \mu(B(\gamma(t_k), \varepsilon r/32)) \\
&\leq C_V C_D^{32[3/4+\delta]\varepsilon^{-1}} \mu(B(o, r/2)) \\
&\to C_V C_D^{24\varepsilon^{-1}} \mu(B(o, r/2)) \qquad \text{as } \delta \to 0
\end{aligned}$$

□

CHAPTER 4

DOUBLING PROPERTY FOR WEIGHTED MEASURES

The goal of this chapter is to answer the question when the weighted measure $\mu = hd\nu$ will satisfy doubling property if ν does. The simplest form of h is a function of distance function i.e. $h(x) = a(d(x, \Sigma))$ for some function $a : [0, \infty) \to [0, \infty]$ and closed set Σ. Follows Grigor'yan and Saloff-Coste's idea, the author will focus to the functions a that will immediately imply doubling condition for Σ-remote balls.

4.1 Remotely Constant Functions

Definition 4.1.1 *A nonzero function $a : [0, \infty) \to [0, \infty]$ is said to be remotely constant if $a(1) < \infty$ and there exists a constant $c = c_a \geq 1$ such that for any $r > 0$,*

$$\sup_{[r,3r]} a \leq c \inf_{[r,3r]} a$$

Note that any reciprocal of remotely constant functions is also remotely constant. This class of functions also closed under finite additions, multiplications, maximum and minimum.

In a sense, a is remotely constant if and only if it is roughly constant on any interval remoted to 0. The condition $a(1) < \infty$ is simply to guarantee that the function a is not infinite anywhere except possibly at 0. This condition is essential if one want the weighted measure $\mu = a(d(\cdot, \Sigma))d\nu$ to be locally finite. It will be shown later that any remotely constant function is equivalent to a continuous function. Moreover, it is equivalent to finitely differentiable functions of any finite order. It can be said that remotely constant functions behave rather well in rough geometry.

Here functions f and g are equivalent, written $f \sim g$, if there exists a constant $c \geq 1$ such that $c^{-1} f \leq g \leq cf$.

Proposition 4.1.2 *For any remotely constant function a, there exists a constant $\beta \geq 0$ and an increasing continuous function $\tilde{a}$ such that $a(r) \sim r^{-\beta}\tilde{a}(r)$.*

Proof. First assume that $\sup_{[r,3r]} a \leq c \inf_{[r,3r]} a$ for all $r > 0$, for some fixed constant $c < 3$. Define $\tilde{a}(r) = \int_0^r a$. Then

$$\begin{aligned} \tilde{a}(r) &= \sum_{i=0}^{\infty} \int_{\frac{r}{3^{i+1}}}^{\frac{r}{3^i}} a \\ &\leq a(r) \sum_{i=0}^{\infty} c^{i+1} \int_{\frac{r}{3^{i+1}}}^{\frac{r}{3^i}} ds \\ &= a(r) r \sum_{i=0}^{\infty} c^{i+1} \Big(\frac{1}{3^i} - \frac{1}{3^{i+1}} \Big) \\ &= \frac{6c}{3(3-c)} a(r) r \end{aligned}$$

By the same argument, one also have $\tilde{a}(r) \geq \frac{2}{3c-1} a(r) r$. For general a, apply the result to $a^{1/k}$ for k big enough so that $c^{1/k} < 3$ gives $(\widetilde{a^{1/k}})^k(r) \sim r^k a(r)$.

□

The above proposition implies that there must exists the best *i.e.* the smallest nonnegative β which is bounded above by $\log_3 c$. However, both are not equal in general. Consider for example $a(r) = \Big(r\chi_{[0,1]} + (5r+1)\chi_{(1,3)} + 2r\chi_{[3,\infty)} \Big)^{-1}$. In this case, $c_a = 6$ but $a(r) \sim r^{-1}$.

Definition 4.1.3 *Let a be a remotely constant function. The decay rate of a is the infimum of all nonnegative β such that there exists a constant $c > 0$ with*

$$c\Big(\frac{r}{s}\Big)^{-\beta} \leq \frac{a(r)}{a(s)} \quad \forall r > s > 0$$

Denote $[\beta]$ the set of all remotely constant functions with decay rate β and (β) the set of those $a \in [\beta]$ that such that there exists a constant $c > 0$ with $c\left(\frac{r}{s}\right)^{\beta} \leq \frac{a(r)}{a(s)} \quad \forall r > s > 0$. Furthermore, denote $(\beta)_s = \{a \in (\beta) : \lim_{r\to 0} r^{\beta} a(r) = 0\}$. For example, (0) is the set of all functions equivalent to nondecreasing functions.

Proposition 4.1.4 *For all $\beta \geq 0$, $\emptyset \subsetneq (\beta)_s \subsetneq (\beta) \subsetneq [\beta]$.*

Proof. The middle inequality is simple. For the first one, consider $r \mapsto r^{-\beta}\ln(1 + r)$. For the last one, let $a(r) = \ln(e + 1/r)$. Clearly, $a \notin (0)$. Denote $b_\lambda(x) = x^{-\lambda}\ln(e + x)$. Since $b'_\lambda < 0$ outside a compact set, b_λ is equivalent to a nonincreasing function. Hence, $r \mapsto r^{\lambda}a(r)$ is equivalent to a nondecreasing function. This directly implies $a \in [0]$. For $\beta > 0$, use $r \mapsto r^{-\beta}\ln(e + 1/r)$ instead.

□

Proposition 4.1.5 *Let a be a remotely constant function with decay rate β_0 and $\beta \neq \beta_0$. Then there exists a nondecreasing continuous function $\tilde{a}$ such that $r^{-\beta}\tilde{a} \sim a$ if and only if $\beta > \beta_0$.*

Proof. If $\beta > \beta_0$, set $\tilde{a} = \sup_{0<s<r} s^{\beta}a(s)$. Since $s^{\beta}a(s) \lesssim r^{\beta}a(r)$ for all $r > s > 0$, $\tilde{a} \sim r^{\beta}a$.

On the contrary, if $r^{-\beta}\tilde{a} \sim a$ for some nondecreasing function $\tilde{a}$, then

$$\frac{a(r)}{a(s)} \sim \Big(\frac{s}{r}\Big)^{\beta}\frac{\tilde{a}(r)}{\tilde{a}(s)} \geq \Big(\frac{s}{r}\Big)^{\beta}$$

Therefore β must be bigger than the decay rate of a.

□

4.2 Doubling Exponent

The goal of this chapter is, after all, to determine to what extent the doubling property of weighted measure will hold. In other words, to determine the biggest β so that all $d\mu = a(d(\cdot, \Sigma))dv, a \in [\beta]$ satisfies doubling property. It turns out that on a large class of Σ, it is sufficient to consider only $a(r) = r^{-\beta}$.

Definition 4.2.1 *Fixed $\rho \in (0, 1]$. A closed subset Σ of a complete length space (X, d) satisfies ρ-*skew condition *if for any $o \in \Sigma$ and $r > 0$, the set*

$$\Sigma_\rho(o, r) = \{x \in X : \rho r \leq d(x, \Sigma) \leq d(x, o) \leq r\}$$

is nonempty. Σ is said to be fully accessible *if it satisfies ρ-skew condition with $\rho = 1$.*

Denote $c\Sigma_\rho(o, r) = \cup_{0 \leq s \leq r} \Sigma_\rho(o, s)$ and $c\Sigma_\rho(o) = \cup_{r>0} c\Sigma_\beta(o, r)$.

Proposition 4.2.2 *Let v be a doubling measure in a metric space (X, d). For any closed, measure zero, subset Σ satisfying ρ-skew condition, and any remotely constant a, the weighted measure $d\mu = a(d(\cdot, \Sigma))dv$ is doubling if and only if there exists a constant $c > 0$ such that*

$$\mu(B(o, r)) \leq ca(r)v(B(o, r))$$

for any $o \in \Sigma$ and $r > 0$.

Proof. First of all, the fact that a is remotely constant immediately implies that μ satisfies doubling property for Σ-remote balls. Moreover, $\mu(B(x,r)) \sim a(d(x,\Sigma))\nu(B(x,r))$ for any remote balls $B(x,r)$. If one can find a constant $c' > 0$ so that $\mu(B(o,r)) \geq c'a(r)\nu(B(o,r))$ for any $o \in \Sigma$ and $r > 0$, then combines this with the original assumption, we get $\mu(B(o,r)) \sim a(r)\nu(B(o,r))$. This implies both doubling property for anchored balls and volume comparison condition. Therefore, μ must be doubling.

Conversely, if μ is doubling, then for each $o \in \Sigma$ and $r > 0$, and $x \in X$ so that $d(x,\Sigma) \geq \rho d(x,o) = \rho r$. Then

$$\begin{aligned} \mu(B(o,r)) &\leq C_V\mu(B(x,\rho\frac{r}{32})) \\ &\sim a(r)\nu(B(x,\rho\frac{r}{32})) \end{aligned}$$

Lastly, lets find the constant c'. This can be shown analogously as the previous argument. For each $o \in \Sigma$ and $r > 0$, pick $x \in \Sigma_\rho(o,r)$. Then

$$\begin{aligned} \mu(B(o,2r)) &\geq \mu(B(x,\frac{\rho r}{2})) \\ &\geq \Big(\inf_{[\frac{\rho r}{2},\frac{3\rho r}{2}]} a\Big)\nu(B(x,\frac{\rho r}{2})) \\ &\sim a(r)\nu(B(o,r)) \end{aligned}$$

□

Now it is time to introduce the main concept in this section.

Definition 4.2.3 *Fixed a doubling measure ν on a metric space (X,d). For any closed measure zero set $\Sigma \subset X$, the doubling exponent $\beta_D(\Sigma)$ of Σ is the supremum of all $\beta \geq 0$ such that the weighted measure $d(\cdot,\Sigma)^{-\beta}d\nu$ is doubling.*

Proposition 4.2.4 *Fixed a doubling measure ν on a metric space (X, d), and closed mesure zero sets $\Sigma_i \subset X$ where $i = 1, \ldots, n$. Then*

$$\beta_D(\Sigma_1 \cup \cdots \cup \Sigma_n) \geq \inf_{1 \leq i \leq n} \beta_D(\Sigma_i)$$

Moreover, if all Σ_i satisfy ρ-skew condition, then

$$\beta_D(\Sigma_1 \cup \cdots \cup \Sigma_n) = \inf_{1 \leq i \leq n} \beta_D(\Sigma_i)$$

Proof. This follows from the fact that

$$\frac{1}{d(\cdot, \Sigma_1 \cup \cdots \cup \Sigma_n)^\beta} \sim \frac{1}{d(\cdot, \Sigma_1)^\beta} + \cdots + \frac{1}{d(\cdot, \Sigma_n)^\beta}$$

for any $\beta \geq 0$. So if $\beta < \inf_{1 \leq i \leq n} \beta_D(\Sigma_i)$, then $d(\cdot, \Sigma_i)^{-\beta} d\nu$ is doubling for all i which implies $d(\cdot, \Sigma_1 \cup \cdots \cup \Sigma_n)^{-\beta} d\nu$ is as well. This proves the first inequality.

Moreover, $d(\cdot, \Sigma_i)^{-\beta} \leq d(\cdot, \Sigma_1 \cup \cdots \cup \Sigma_n)^{-\beta}$. Combining this with the assumption that all Σ_i satisfy ρ-skew condition, then the equality must follows.

□

Proposition 4.2.5 *For any doubling measure ν on a metric space (X, d) and any closed mesure zero subsets $\Sigma_1 \subset \Sigma_2$ of X, $\beta_D(\Sigma_1) \geq \beta_D(\Sigma_2)$ provided that Σ_2 satisfies ρ-skew condition.*

Proof. First note that Σ_1 also satisfies ρ-skew condition. Also for any $\beta < \beta_D(\Sigma_2)$ $o \in \Sigma_1$, and $r > 0$

$$\begin{aligned} \int_{B(o,r)} \frac{1}{d(x, \Sigma_1)^\beta} d\nu(x) &\leq \int_{B(o,r)} \frac{1}{d(x, \Sigma_2)^\beta} d\nu(x) \\ &\lesssim \frac{1}{r^\beta} \nu(B(o, r)) \end{aligned}$$

and the result follows.

$\square$

Corollary 4.2.6 *Let ν be any doubling measure on a metric space (X, d) and $\Sigma_1 \subset \Sigma_2$ be any closed mesure zero subsets of X such that $\beta_D(\Sigma_1) = \beta_D(\Sigma_2)$. Assume that Σ_2 satisfies ρ-skew condition, then for any $\Sigma_1 \subset \Sigma \subset \Sigma_2$, $\beta_D(\Sigma) = \beta_D(\Sigma_1)$.*

Theorem 4.2.7 *Let ν be a doubling measure in a geodesic space. For any $\beta \geq 0$ and a closed mesure zero set Σ satisfies ρ-skew condition, the following are equivalent.*

(a) $\beta < \beta_D(\Sigma)$

(b) *$a(d(\cdot, \Sigma))d\nu$ is doubling for all $a \in [\beta]$, and $\beta \neq \beta_D(\Sigma)$*

(c) *$a(d(\cdot, \Sigma))d\nu$ is doubling for all $a \in (\beta)$, and $\beta \neq \beta_D(\Sigma)$*

(d) *$a(d(\cdot, \Sigma))d\nu$ is doubling for all $a \in (\beta)_s$, and $\beta \neq \beta_D(\Sigma)$*

Proof. It is sufficient to prove (a) implies (b) and (d) implies (a). Assume that (a) holds. If one can show that $d(\cdot, \Sigma)^\beta d\nu$ is doubling, then for any nondecreasing function $\tilde{a}$,

$$\begin{aligned} \int_{B(o,r)} \frac{\tilde{a}(d(x,\Sigma))}{d(x,\Sigma)^\beta} d\nu(x) &\leq \tilde{a}(r) \int_{B(o,r)} \frac{1}{d(x,\Sigma)^\beta} d\nu(x) \\ &\lesssim \tilde{a}(r) \frac{1}{r^\beta} \nu(B(o,r)) \end{aligned}$$

Therefore $r \mapsto \tilde{a}(r) r^{-\beta}$ must also be doubling. Therefore $a(d(\cdot, \Sigma))d\nu$ is doubling for all $a \in [\beta]$. This show that it is sufficient to prove $d(\cdot, \Sigma)^\beta d\nu$ is doubling.

By definition, there exists $\beta_0 > \beta$ such that $d(\cdot, \Sigma)^{\beta_0} d\nu$ is doubling. By the same argument, one would have

$$\int_{B(o,r)} \frac{1}{d(x,\Sigma)^\beta} d\nu(x) = \int_{B(o,r)} \frac{d(x,\Sigma)^{\beta_0 - \beta}}{d(x,\Sigma)^{\beta_0}} d\nu(x)$$

$$
\begin{aligned}
&\leq \; r^{\beta_0-\beta}\int_{B(o,r)}\frac{1}{d(x,\Sigma)^{\beta_0}}d\nu(x) \\
&\lesssim \; r^{\beta_0-\beta}\frac{1}{r^{\beta_0}}\nu(B(o,r)) \\
&= \; r^{\beta}\nu(B(o,r))
\end{aligned}
$$

for any $o \in \Sigma$ and $r > 0$.

Now, lets prove (d) implies (a). Note that for all $r > s > 0$,

$$\frac{\ln(1+r)}{\ln(1+s)} \leq \frac{r}{s}$$

Therefore $r \mapsto \ln(1+r)$ is remotely constant. If there exists $\delta > 0$ such that for some $c > 0$,

$$\frac{\ln(1+r)}{\ln(1+s)} \geq c\Big(\frac{r}{s}\Big)^{\delta}, \qquad \forall r > s > 0$$

Then $c\frac{\ln(1+s)}{s^{\delta}} \leq \frac{\ln(1+r)}{r^{\delta}} \to 0$ as $r \to \infty$ which leads to a contradiction. This implies that $r \mapsto \ln(1+r) \in (0)_s$.

Define $a_\delta(r) = r^{-\beta}[\ln(1+r)]^{\delta}$. It follows that $a_\delta \in (\beta)_s$. Hence for any $o \in \Sigma$ and $r > 0$,

$$
\begin{aligned}
\frac{[\ln(1+r)]^{\delta}}{r^{\delta}}\int_{B(o,r)}\frac{1}{d(x,\Sigma)^{\beta-\delta}}d\nu(x) \;&\leq\; \int_{B(o,r)}\frac{[\ln(1+r)]^{\delta}}{d(x,\Sigma)^{\beta}}d\nu(x) \\
&\lesssim\; \frac{[\ln(1+r)]^{\delta}}{r^{\beta}}\nu(B(o,r))
\end{aligned}
$$

Thus, $\int_{B(o,r)}\frac{1}{d(x,\Sigma)^{\beta-\delta}}d\nu \lesssim \frac{1}{r^{\beta-\delta}}\nu(B(o,r))$ which directly implies $\beta - \delta \leq \beta_D(\Sigma)$. Since this holds for any $\delta > 0$, $\beta \leq \beta_D(\Sigma)$.

□

Now is the time to face the real question, how can one compute $\beta_D(\Sigma)$? In the beginning, the author shows that the doubling exponent of any affine subspace of a Euclidean space with respect to Lebesgue measure is its codimension. The next section show similar result for singleton. For a more general case, however, is not always true.

4.2.1 Doubling Exponent of Singleton

At this point, it should not be surprise that the doubling exponent of singleton is the growth rate of doubling measures.

Theorem 4.2.8 *Let ν be a doubling measure in a geodesic space (X, d) and $o \in X$. The doubling exponent $\beta_D(o)$ of $\{o\}$ is the supremum of all $\beta \geq 0$ such that there exists $c > 0$ so that*

$$\frac{\nu(B(o,r))}{\nu(B(o,s))} \geq c\Big(\frac{r}{s}\Big)^{\beta} \quad \forall r > s > 0$$

Moreover, the measure $d(\cdot, o)^{-\beta_D(o)} d\nu$ is not doubling.

Proof. For convenient, denote the supremum of such β as β_0. Fixed $\beta > 0$ and set $d\mu = d(\cdot, o)^{-\beta} d\nu$. If μ is doubling, then there must exists $\epsilon, c > 0$ such that

$$c\Big(\frac{r}{s}\Big)^{\epsilon} \leq \frac{\mu(B(o,r))}{\mu(B(o,s))}, \quad \forall r > s > 0$$

On the other hand, $\mu(B(o,r)) \sim r^{-\beta}\nu(B(o,r))$. This implies that $\beta + \epsilon \leq \beta_0$ i.e $\beta < \beta_0$.

On the contrary, if $\beta < \beta_0$, then one can choose $\beta' \in (\beta, \beta_0)$. Now,

$$\begin{aligned}
\mu(B(o,r)) &= \sum_{i=0}^{\infty} \int_{B(o,\frac{r}{3^i}) - B(o,\frac{r}{3^{i+1}})} \frac{1}{d(x,o)^{\beta}} d\nu(y) \\
&\leq \sum_{i=0}^{\infty} \frac{1}{r^{\beta}} 3^{i\beta} (\nu(B(o, \frac{r}{3^i})) - \nu(B(o, \frac{r}{3^{i+1}}))
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{r^\beta}\sum_{i=0}^{\infty} 3^{i\beta}\nu(B(o,\frac{r}{3^i}))\\
&\lesssim \frac{1}{r^\beta}\sum_{i=0}^{\infty} 3^{i\beta}\frac{\nu(B(o,r))}{3^{i\beta'}}\\
&\leq \frac{3^{\beta'-\beta}}{3^{\beta'-\beta}-1}\frac{\nu(B(o,r))}{r^\beta}
\end{aligned}
$$

This shows that μ is doubling.

$\square$

Corollary 4.2.9 *Let ν be a doubling measure in a geodesic space (X,d) and $o \in X$. Denote $\alpha_D(o)$ the supremum of all $\alpha \geq 0$ such that there exists $c > 0$ so that*

$$\frac{\nu(B(o,r))}{\nu(B(o,s))} \geq c\Big(\frac{r}{s}\Big)^{\alpha} \quad \forall r > s > 0$$

Then the following are equivalent.

(a) $\beta < \alpha_D(\Sigma)$.

(b) *$a(d(\cdot,\Sigma))d\nu$ is doubling for all $a \in [\beta]$.*

(c) *$a(d(\cdot,\Sigma))d\nu$ is doubling for all $a \in (\beta)$.*

(d) *$a(d(\cdot,\Sigma))d\nu$ is doubling for all $a \in (\beta)_s$.*

4.2.2 Assouad Dimension revisited

In this section, the author compute the bounds of doubling exponent on a class of Σ. As in the Euclidean case, the bounds are related to Assouad dimension of Σ. Actually, it is more related to the change in Assouad dimension.

Definition 4.2.10 *A doubling space (X, d) is said to have consistent Assouad dimension $\alpha_A = \alpha_A(X)$ if for each $\alpha > \alpha_A$, there exists a constant $c > 0$ such that for any $r > s > 0$, $x \in X$, and $y \in B(x, r)$, one can find ϵ-sets $S_\epsilon(x, r)$ of $B(x, r)$ and $S_\epsilon(y, s)$ of $B(y, s)$ such that*

$$\limsup_{\epsilon \to 0} \frac{\#S_\epsilon(y, s)}{\#S_\epsilon(x, r)} \geq c\left(\frac{s}{r}\right)^\alpha$$

where $\#A$ denote the number of elements of A.

Recall that an ϵ-set of a subset B of a metric space is any maximal subset of B such that each elements are at least ϵ distance to each others. By Proposition 3.1.6, the Assouad dimension is always smaller than the consistent Assouad dimension. At this moment, the author do not yet know whether these two numbers are the same.

First the author shows that it actually do not matter which ϵ-set to choose from as long as one adjusts the constant c appropriately. It also implies that one may replace $\limsup$ in the definition of consistent Assouad dimension by $\liminf$.

Lemma 4.2.11 *Let (X, d) be a doubling space. Then there exists a constant $N \geq 1$ such that for any $x \in X$, $r > 0$, and any two ϵ-set S, S' of $B(x, r)$,*

$$\frac{1}{N}\#S \leq \#S' \leq N\#S$$

Proof. By definition, $B(y, \epsilon), y \in S$ is a covering of $B(x, r)$. Denote $S_y = \{z \in S' : z \in B(y, \epsilon)\}$. Since X is a doubling space, there is a number N depends solely on X such that $\#S_y \leq N$. This implies

$$\#S' \leq \sum_{y \in S} \#S_y \leq N\#S$$

By switching S and S', one also have $\#S \leq N\#S'$.

□

Proposition 4.2.12 *Any postcritically finite fractal(e.g. [42]) has consistent Assouad dimension equal to its Hausdorff dimension.*

Proof. For any open subsets $U \subset V$ of a doubling space X, one have $\dim_A U \leq \dim_A V$. The fact that X is a postcritically finite fractal then forces $\dim_A U = \dim_A V$. Moreover, we know that $\dim_A X$ equal to Hausdorff dimension in this case. Therefore, X must have consistent Assouad dimension.

□

Definition 4.2.13 *Let ν be a doubling measure in a geodesic space (X, d) and $\Sigma \subset X$. The uniform growth rate $\alpha_D = \alpha_D(\Sigma)$ over Σ is the supremum of all $\alpha > 0$ such that there exists $c > 0$ with*

$$\frac{\nu(B(o, r))}{\nu(B(o, s))} \geq c\Big(\frac{r}{s}\Big)^{\alpha}$$

for any $r > s > 0$ and $o \in \Sigma$.

Recall that if (X, d) is a Euclidean space, ν is a Lebesgue measure, and Σ be its affine subspace, then $\beta_D(\Sigma) = \dim X - \dim \Sigma = \alpha_D - \alpha_A$ as shown in Chapter 2. So one might ask whether this is always true or not. The answer is no and it should not be surprising. Consider for example $d\nu = |x|^{-n+1}dx$ on the Euclidean space of dimension n. In this case $\alpha_D(\mathbb{R}^{n-1}) = 1$ while $\alpha_A(\mathbb{R}^{n-1}) = n - 1$. This happens simply because the measure is not comparable at different points. This leads the author to prove the following result.

Theorem 4.2.14 *Let ν be a doubling measure in a geodesic space (X, d) and $\Sigma \subset X$ satisfying ρ-skew condition and have consistent Assouad dimension α_A. Then*

$$\beta_D(\Sigma) \geq \alpha_D(\Sigma) - \alpha_A(\Sigma)$$

Proof. If $\alpha_D - \alpha_A \leq 0$, then the above inequality becomes trivial. So it is natural to assume that $\alpha_D - \alpha_A > 0$.

Fixed $o \in \Sigma$, $r > 0$, and $\beta < \alpha_D$. Then there exists $c > 0$ and $\beta_0 > \beta$ such that for any $o' \in \Sigma$ with $d(o, o') < r$,

$$\frac{\nu(B(o', s))}{\nu(B(o', t))} \geq c\Big(\frac{s}{t}\Big)^{\beta_0}$$

It follows that for such o',

$$\begin{aligned} \int_{c\Sigma_\rho(o',r)} \frac{1}{d(\cdot, o')^\beta} d\nu &\leq \int_{B(o',r)} \frac{1}{d(\cdot, o')^\beta} d\nu \\ &\lesssim \frac{1}{r^\beta} \nu(B(o', r)) \\ &\lesssim \frac{1}{r^\beta} \nu(B(o, r)) \end{aligned}$$

uniformly. Here ρ is small but fixed, say $\rho < 1/4$. Since on $c\Sigma_\rho(o', r)$, $d(\cdot, o') \sim d(\cdot, \Sigma)$, it follows that

$$\int_{c\Sigma_\rho(o',r)} \frac{1}{d(\cdot, \Sigma)^\beta} d\nu \lesssim \frac{1}{r^\beta} \nu(B(o, r)) \tag{4.1}$$

uniformly as well.

Next, fixed $\alpha > \alpha_A$. Choose $c > 0$, $\epsilon_n \searrow 0$, and ϵ-sets $S_n(o, r) = S_{\epsilon_n}(o, 2r) \subset S_{n+1}$ such that

$$\lim_{n\to\infty} \frac{\# S_{\epsilon_n}(o', s)}{\# S_n(o, r)} \geq c\Big(\frac{s}{r}\Big)^\alpha$$

Let $S_n(o', s)$ be the set of all $y \in S_n(o, r)$ such that $B(y, \epsilon_n) \cap B(o', s) \neq \emptyset$. It can be proved the same way as before that $\#S_n(o', s) \gtrsim \#S_{\epsilon_n}(o', s)$. Therefore, one also have

$$\liminf_{n\to\infty} \frac{\#S_n(o', s)}{\#S_n(o, r)} \gtrsim \left(\frac{s}{r}\right)^{\alpha}$$

Fixed $\delta = \frac{1}{\rho} - 2 > 0$. For each $x \in B(o, r)$, choose $o_x \in \Sigma$ such that $d(x, \Sigma) = d(x, o_x) = r_x$. Let n be big enough so that $\epsilon_n < r_x$, then for any $o' \in S_n(o_x, \delta r_x)$,

$$\begin{aligned} d(x, o') &\leq d(x, o_x) + d(o_x, o') \\ &\leq r_x + \delta r_x + \epsilon_n \\ &\leq (2 + \delta) r_x \end{aligned}$$

Therefore, $x \in \Sigma_\rho(o', r)$. This implies

$$\begin{aligned} \liminf_{n\to\infty} \frac{1}{\#S_n(o, r)} \sum_{o' \in S_n(o,r)} \chi_{c\Sigma_\rho(o',r)}(x) &\geq \liminf_{n\to\infty} \frac{1}{\#S_n(o, r)} \#S_n(o_x, \delta r_x) \\ &\gtrsim \left(\frac{r_x}{r}\right)^{\alpha} \\ &= \left(\frac{d(x, \Sigma)}{r}\right)^{\alpha} \end{aligned}$$

for any $x \in B(o, r)$. Combining this with equation 4.1 and get

$$\begin{aligned} \frac{1}{r^\beta} \nu(B(o, r)) &\gtrsim \liminf_{n\to\infty} \int_{B(o,r)} \Big(\frac{1}{\#S_n(o, r)} \sum_{o' \in S_n(o,r)} \chi_{c\Sigma_\rho(o',r)}(x) \Big) \frac{1}{d(x, \Sigma)^\beta} d\nu(x) \\ &\gtrsim \int_{B(o,r)} \frac{1}{r^\alpha d(x, \Sigma)^{\beta-\alpha}} d\nu(x) \end{aligned}$$

This is equivalent to $\int_{B(o,r)} \frac{1}{d(x,\Sigma)^{\beta-\alpha}} d\nu(x) \lesssim \frac{1}{r^{\beta-\alpha}} \nu(B(o, r))$.

Since $\beta < \alpha_D$ and $\alpha > \alpha_A$ are arbitrary, it follows that

$$\int_{B(o,r)} \frac{1}{d(x, \Sigma)^\beta} d\nu(x) \lesssim \frac{1}{r^\beta} \nu(B(o, r))$$

for any $\beta < \alpha_D - \alpha_A$.

$\square$

Of course, if the measure are comparable at different points in Σ, then $\alpha_D \geq \alpha_A$. The proof is based on following lemma.

Lemma 4.2.15 *Let ν be a doubling measure on (X, d) and $\Sigma \subset X$ having consistent Assouad dimension $\alpha_A(\Sigma)$. Assume that there exists $c \geq 1$ such that for any $o, o' \in \Sigma$ and $r > 0$ with $r \leq d(o, o')$,*

$$\frac{1}{c}\nu(B(o,r)) \leq \nu(B(o',r)) \leq c\nu(B(o,r))$$

then $\alpha_A(\Sigma)$ is at most the infimum of all $\alpha > 0$ such that there exists $c > 0$ so that

$$\frac{\nu(B(o,r))}{\nu(B(o,s))} \leq c\Big(\frac{r}{s}\Big)^{\alpha}$$

for any $o \in \Sigma$ and $r > s > 0$.

Proof. Fixed $\alpha > \alpha_D$ and $o \in \Sigma$ and $r > s > 0$. For any ϵ-sets $S_\epsilon(o,r)$ and $S_\epsilon(o,s)$, $\nu(B(o, r+\epsilon_n)) \geq \nu(B(o, \frac{\epsilon_n}{2}))\#S_n(o,r)$ while $\nu(B(o,s)) \leq \nu(B(o,\epsilon_n))\#S_n(o,s)$. Therefore,

$$\begin{aligned}
\Big(\frac{r}{s}\Big)^{\alpha} &\gtrsim \liminf_{\epsilon\to 0} \frac{\nu(B(o, r+\epsilon_n))}{\nu(B(o,s))} \\
&\geq \liminf_{\epsilon\to 0} \frac{\nu(B(o,\frac{\epsilon_n}{2}))\#S_n(o,r)}{\nu(B(o,\epsilon_n))\#S_n(o,s)} \\
&\gtrsim \liminf_{\epsilon\to 0} \frac{\#S_n(o,r)}{\#S_n(o,s)}
\end{aligned}$$

$\square$

Theorem 4.2.16 *Let ν be a doubling measure on (X, d) and $\Sigma \subset X$ having consistent Assouad dimension $\alpha_A(\Sigma)$. Assume that*

$$\frac{\nu(B(o,r))}{\nu(B(o,s))} \sim \left(\frac{r}{s}\right)^{\alpha_D}$$

uniformly in $o \in \Sigma$ and $r > s > 0$. Then $\alpha_A \leq \alpha_D$.

Proof. Combining the above assumption with the fact that $\nu(B(o,r)) \sim \nu(B(o',r))$ for any $d(o,o') \leq r$, one also have $\nu(B(o,s)) \sim \nu(B(o',s))$ for any $d(o,o') \leq r$ and $s \leq r$. From the previous Lemma, $\alpha_A \leq \alpha_D$.

□

Conjecture 4.2.1 *Let ν be a doubling measure on (X, d) and $\Sigma \subset X$ having consistent Assouad dimension $\alpha_A(\Sigma)$. Assume that*

$$\frac{\nu(B(o,r))}{\nu(B(o,s))} \sim \left(\frac{r}{s}\right)^{\alpha_D}$$

for any $o \in \Sigma$ and $r > 0$. Then $\beta_D = \alpha_D - \alpha_A$.

4.3 Examples

This section collects some simple examples to demonstrate the computation power of all that have been done so far. Most of them will be Euclidean spaces but the same idea can be more generally applied as well.

Example 4.3.1 *Let ν be the Lebesgue measure in the Euclidean space and Σ be the closer of an open subset of its affine subspace with codimension k. It is not hard to see that* *its consistent Assouad dimension is the dimension of the affine subspace containing it. Using Theorem 4.2.14, $\beta_D(\Sigma) \geq k$. Then the fact that doubling property must be locally integrable forces $\beta_D(\Sigma) = k$.*

Example 4.3.2 *Let ν be the Lebesgue measure in the Euclidean space and $\Sigma = \mathbb{Z}^k \times \{0\}^{n-k}$ be its subset. Again, Theorem 4.2.14, $\beta_D(\Sigma) \geq n - k$. Moreover,*

$$\begin{aligned} \int_{B(o,r)} \frac{1}{d(\cdot,\Sigma)^\beta} d\nu &\sim n^k \int_{B(o,1)} \frac{1}{d(\cdot,\Sigma)^\beta} d\nu + \int_{B(o,r)-\{x:d(x,\Sigma)<1\}} \frac{1}{d(\cdot,\Sigma)^\beta} d\nu \\ &\sim r^k + r^{n-\beta} \end{aligned}$$

for any big $r > 0$. Therefore, doubling property only holds when $k \leq n - \beta$ and hence $\beta_D(\Sigma) = n - k$.

Note that even though both $\mathbb{Z}^k \times \{0\}^{n-k}$ and $\mathbb{R}^k \times \{0\}^{n-k}$ have the same doubling exponent $n - k$, their behavior at $n - k$ are different. The weighted measure $d(\cdot, \mathbb{Z}^k \times \{0\}^{n-k})^{n-k} d\nu$ is doubling but the weighted measure $d(\cdot, \mathbb{R}^k \times \{0\}^{n-k})^{n-k} d\nu$ is not. The latter is not even locally integrable.

Example 4.3.3 *Let ν be the Lebesgue measure in the Euclidean space and Σ be an ϵ-set of an affine subspace $\mathbb{R}^k \times \{0\}^{n-k}$. Then $\beta_D(\Sigma) = n - k$. The proof is similar to the previous example.*

Example 4.3.4 *Let ν be the Lebesgue measure in the Euclidean space and Σ be a set containing an ϵ-set of an affine subspace $\mathbb{R}^k \times \{0\}^{n-k}$ containing Σ. Then $\beta_D(\Sigma) = n - k$. The is an immediate fact of Corollary 4.2.6.*

Example 4.3.5 *Let ν be the Lebesgue measure in the Euclidean space of dimension $n > 1$, and Σ be a finite union of rays originated from the origin. Then $\beta_D(\Sigma) = n - 1$. The is also an immediate fact of Corollary 4.2.6.*

Example 4.3.6 *Let ν be the Lebesgue measure in the Euclidean space of dimension $n > 1$, and Σ be its compact submanifolds, possibly with boundary, of dimension $k < n$. It is not hard to see that $\alpha_A = k$, so $\beta_D \geq n - k$. Locally integrability then forces $\beta_D = n - k$.*

Note that the compactness condition can also be replaces by the boundedness and nonnegativeness of curvature. The idea is that this submanifold must be quasi-isometric to the affine subspace.

Example 4.3.7 *Let ν be the Lebesgue measure in the Euclidean space of dimension $n > 1$, and Σ be a finite complex with dimension $k < n$. Then $\beta_D = n - k$. This follows from Corollary 4.2.6 and local integrability condition.*

Example 4.3.8 *Let ν be the Lebesgue measure in the Euclidean space $\mathbb{R}^3$ and $\Sigma = \mathbb{S}^1 \times \mathbb{R}$ where $\mathbb{S}^1$ is the unit circle in $\mathbb{R}^2$. In this case $\beta_D = \beta_D(\mathbb{S}^1) = 1$.*

On the contrary, if $\Sigma' = \mathbb{S}^1 \times \mathbb{Z}$, then $\alpha_A = 1$. Therefore $\beta_D = 2$. In contrary to earlier example, Σ' contains a 1-set of Σ, yet $\beta_D(\Sigma') \neq \beta_D(\Sigma)$.

What about the discrete set $\Sigma'' = \{e^{ik/n} : k = 1, \ldots, n\} \times \mathbb{Z}$? Is $\beta_D(\Sigma'') = 3$ in this case? The answer is no. The doubling exponent $\beta_D(\Sigma'')$ is still 2.

What happen in this example is that on a large scale, all these sets behave like a one dimensional space, while locally they behaves different. It then follows that the doubling exponent can never exceed $3 - 1 = 2$.

CHAPTER 5

DIRICHLET SPACES

This chapter introduces Dirichlet forms on general metric spaces focusing on the strongly local ones. For more information regarding general theory of Dirichlet forms, see for example [26] and [6].

5.1 Dirichlet Spaces

A Dirichlet form is a positive symmetric bilinear form with some special properties, so the author will start by reviewing the definition of symmetric bilinear forms.

A densely defined positive symmetric bilinear form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on a Hilbert space $\mathbf{H}$ is said to be closed if its domain $\mathcal{D}(\mathcal{E})$ is a Hilbert space under the Dirichlet inner product

$$\langle f, g\rangle_{\mathcal{E}} := \langle f, g\rangle_{\mathbf{H}} + \mathcal{E}(f, g)$$

It is said to be closable if it has a closed extension. The smallest closed extension of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is called the closure of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. It is easy to show that a densely defined bilinear form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is closable if and only if for any Cauchy sequence (f_k) in $\mathcal{D}(\mathcal{E})$ with $f_k \to 0$ in $\mathbf{H}$, $f_k \to 0$ in $\mathcal{D}(\mathcal{E})$.

One way to construct positive symmetric bilinear forms is via the formula $\mathcal{E}(f, g) = \langle Lf, g\rangle$ for some operator L.

Corollary 5.1.1 *Let $L : \mathcal{D}(L) \subset \mathbf{H} \to \mathbf{H}$ be a densely defined operator with the following properties:*

(a) *Positivity:* $\langle Lf, f\rangle \geq 0$*, for all* $f \in \mathcal{D}(L)$*,*

(b) *Symmetry:* $\langle Lf, g\rangle = \langle f, Lg\rangle$ *for all* $f, g \in \mathcal{D}(L)$*,*

(c) *Closability:* $Lf_k \to 0$ *for any* $f_k \in \mathcal{D}(L)$ *converging to* 0 *in* $\mathbf{H}$.

Define $\mathcal{E}(f, g) = \langle Lf, g\rangle$ *with domain* $\mathcal{D}(L)$*. Then* $(\mathcal{E}, \mathcal{D}(L))$ *is closable.*

Conversely, all positive symmetric bilinear forms can be obtained this way. There always exists a unique positive self-adjoint operator L on $\mathbf{H}$ with domain

$$\mathcal{D}(L) := \{h \in \mathbf{H} : \mathcal{E}(h, g) \leq C\|g\|, \ \ \forall g \in \mathcal{D}(\mathcal{E}),\ \exists C > 0\}$$

so that $\mathcal{E}(f, g) = \langle Lf, g\rangle$ for all $f, g \in \mathcal{D}(L)$ and $\mathcal{D}(\mathcal{E}) = \mathcal{D}(L^{1/2})$.

The Hille-Yosida Theorem state that there is one-one corresponse between a positive self-adjoint operator, one parameter semigroup and resolvent. So the above theorem also state that there is a one-one corresponse between a closed positive symmetric bilinear form, one-parameter semigroup, and resolvent. The next two theorems state explicitly how they are related.

Theorem 5.1.2 *Let* (T_t) *be a semigroup of linear operator on* $\mathbf{H}$ *such that*

1. *each* T_t *is a contraction:* $\langle T_t f, T_t f\rangle \leq \langle f, f\rangle, \ \forall f \in \mathbf{H}$*,*
2. *each* T_t *is self-adjoint:* $\langle T_t f, g\rangle = \langle f, T_t g\rangle, \ \ forall f, g \in \mathbf{H}$*,*
3. (T_t) *is strongly continuous:* $T_t f \to f$ *in* $\mathbf{H}$ *as* $t \to 0$ *for all* $f \in \mathcal{H}$*.*

Define $\mathcal{E}(f, g) = \lim_{t\to 0} \langle \frac{f - T_t f}{t}, g\rangle$ *whenever the limit exists and* $\mathcal{D}(\mathcal{E}) = \{f \in \mathbf{H} : \lim_{t\to 0} \langle \frac{f - T_t f}{t}, f\rangle$ *exists*$\}$*. Then* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *is a closed positive symmetric bilinear form. Moreover, all closed positive symmetric bilinear form can be constructed in this way.*

Theorem 5.1.3 *Let (G_α) be a resolvent i.e. (G_α) satisfies the following conditions.*

1. *αG_α is a contraction for all $\alpha > 0$.*
2. *each G_α is self-adjoint.*
3. *(G_α) satisfies resolvent equation: $G_\alpha - G_\beta = (\beta - \alpha)G_\alpha G_\beta$.*
4. *(G_α) is strongly continuous: $\alpha G_\alpha f \to f$ in $\mathbf{H}$ as $\alpha \to \infty$ for all $f \in \mathcal{H}$.*

Define $\mathcal{E}(f, g) = \lim_{\alpha\to\infty} \alpha\langle f - \alpha G_\alpha f, g\rangle$ whenever the limit exists and $\mathcal{D}(\mathcal{E}) = \{f \in \mathbf{H} : \lim_{\alpha\to\infty} \alpha\langle f - \alpha G_\alpha f, f\rangle$ exists$\}$. Then $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a closed positive symmetric bilinear form. All closed positive symmetric bilinear form can be constructed in this way. Moreover,

$$\mathcal{E}(G_\alpha f, g) + \alpha\langle G_\alpha f, g\rangle = \langle f, g\rangle$$

for all $f \in \mathbf{H}, g \in \mathcal{D}(\mathcal{E})$.

5.1.1 Dirichlet Forms

From now on a topological space means a locally compact, second-countable, Hausdorff topological space. Even though Dirichlet forms generally defined in more general topological spaces, these assumptions are what make the analysis possible.

Definition 5.1.4 *Let X be a topological space and μ be a Borel measure on X. A Dirichlet form is a closed positive symmetric bilinear form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(X, d\mu)$ with the following property:*

$$\forall f \in \mathcal{D}(E),\ \ g = (f\vee 0)\wedge 1 \in \mathcal{D}(\mathcal{E}),\ \text{and}\ \mathcal{E}(g, g) \le \mathcal{E}(f, f)$$

The Hilbert space $\mathcal{D}(\mathcal{E})$ is called the Dirichlet space and its norm associated to its inner product $\langle\cdot,\cdot\rangle_{L^2(X,d\mu)} + \mathcal{E}(\cdot,\cdot)$ is referred to as the Dirichlet norm.

By linearity, it is easy to see that $|f|$, $f\vee g$, $f\wedge g$, $f\vee c$, $f\wedge c \in \mathcal{D}(\mathcal{E})$ for any $f, g \in \mathcal{D}(\mathcal{E})$ and $c \in \mathbb{R}$. As for the semigroup associated to it, it turn out to be the submarkovian semigroup.

Proposition 5.1.5 *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a closed positive symmetric bilinear map on $L^2(X, d\mu)$, (T_t) be its associated semigroup, and (G_α) be its associated resolvent. The following properties are equivalent.*

1. (αG_α) *is submarkovian i.e.* $\alpha G_\alpha f \leq 1$ *for all* $f \in L^2(X, d\mu)$ *with* $0 \leq f \leq 1$.
2. (T_t) *is submarkovian i.e.* $T_t f \leq 1$ *for all* $f \in L^2(X, d\mu)$ *with* $0 \leq f \leq 1$.
3. $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *is a Dirichlet form.*
4. *For any* $f \in \mathcal{D}(\mathcal{E})$ *and* $g \in L^2(X, d\mu)$ *such that* $|g| \leq |f|$ *and* $|g(x) - g(y)| \leq |f(x) - f(y)|$ *for all* $x, y \in X$, $g \in \mathcal{D}(\mathcal{E})$ *and* $\mathcal{E}(g, g) \leq \mathcal{E}(f, f)$. *Note that such* g *is called a normal contraction of* f.
5. *For any* $\varepsilon > 0$, *there exists a nondecreasing nonexpansive map* $\phi_\varepsilon : \mathbb{R} \to [-\varepsilon, 1 + \varepsilon]$ *such that* ϕ_ε *is an identity on* $[0, 1]$, *and for all* $f \in \mathcal{D}(\mathcal{E})$, $\phi_\varepsilon(f) \in \mathcal{D}(\mathcal{E})$ *with* $\mathcal{E}(\phi_\varepsilon(f), \phi_\varepsilon(f)) \leq \mathcal{E}(f, f)$.

5.1.2 Energy Measures

It turn out that one can view a regular Dirichlet form as a measure-valued bilinear form. Here regular means that it poses a core. A core C of a Dirichlet form

$(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a subset of $\mathcal{D}(\mathcal{E}) \cap C_c(X)$ such that C is dense in $\mathcal{D}(\mathcal{E})$ under Dirichlet norm and dense in $C_c(X)$ under supremum norm. If $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is regular, then $\mathcal{D}(\mathcal{E}) \cap C_c(X)$ is a core of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a regular Dirichlet form. There exists a unique signed-measure-valued bilinear form Γ, called **energy measure** of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, with domain $\mathcal{D}(\mathcal{E})$ such that

$$\int \phi d\Gamma(f, g) = \frac{1}{2}[\mathcal{E}(f, \phi g) + \mathcal{E}(g, \phi f) - \mathcal{E}(fg, \phi)]$$

for all $\phi \in \mathcal{D}(\mathcal{E}) \cap C_c(X)$, $f, g \in \mathcal{D}(\mathcal{E})$. Moreover, $\Gamma(f, f)$ is actually a finite measure for all $f \in \mathcal{D}(\mathcal{E})$.

The proof of this is based on the fact that for any $f \in \mathcal{D}(\mathcal{E})$ and $\phi \in \mathcal{D}(\mathcal{E}) \cap L^\infty(d\mu)$,

$$\mathcal{E}(f, \phi f) - \frac{1}{2}\mathcal{E}(f^2, \phi) \leq ||\phi||_\infty \mathcal{E}(f, f)$$

One nice thing about Γ is that one can prove Cauchy-Schwarz inequality:

$$\begin{aligned} \int \phi\psi d\Gamma(f, g) &\leq [\int \phi^2 d\Gamma(f, f) \int \psi^2 d\Gamma(g, g)]^{1/2} \\ &\leq \frac{1}{2}[\int \phi^2 d\Gamma(f, f) + \int \psi^2 d\Gamma(g, g)] \end{aligned}$$

for any $f, g \in \mathcal{D}(\mathcal{E})$ and $\phi, \psi \in L^\infty(d\mu)$. The prove is similar to other forms of Cauchy-Schwarz inequality.

5.1.3 Strong Locality

There are three ways to state strong locality, one is its definition, other two are Leibnitz rule and chain rule.

Theorem 5.1.6 (e.g. [45, 46, 47]) *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a regular Dirichlet form on some $L^2(X, d\mu)$ with associated energy measure Γ. The following properties of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ are equivalent.*

1. *Strong locality: for any $f, g \in \mathcal{D}(\mathcal{E})$ with $(f - a)g = 0$ for some constant $a \in \mathbb{R}$, $\mathcal{E}(f, g) = 0$.*
2. *Leibnitz rule: for any $f, g, h \in \mathcal{D}(\mathcal{E})$, $d\Gamma(fg, h) = fd\Gamma(g, h) + gd\Gamma(f, h)$.*
3. *Chain rule: for any $f, g \in \mathcal{D}(\mathcal{E}) \cap L^\infty(X, d\mu)$, and $\eta : \mathbb{R} \to \mathbb{R}$ continuously differentiable, $d\Gamma(\eta(f), g) = \eta'(f)d\Gamma(f, g)$*

The condition $f, g \in L^\infty(x, d\mu)$ in 3. is redundant in the sense that the chain rule still hold without this assumption, however, we only know that $\eta(f)$ is in the local domain instead of the actual domain $\mathcal{D}(\mathcal{E})$.

The local domain $\mathcal{D}_{loc}(\mathcal{E})$ of a strongly local Dirichlet form is defined to be the vector space of all locally square integrable function f such that for any relatively compact open set V, one can find a function $g \in \mathcal{D}(\mathcal{E})$ such that $f = g$ on V. For such f, one can define $d\Gamma(f, f) = d\Gamma(g, g)$ on V. This is well-defined by strong locality. Of course, the formula extends to any $f, h \in \mathcal{D}_{loc}(E)$ by polarization $d\Gamma(f, h) = [d\Gamma(f + h, f + h) - d\Gamma(f - h, f - h)]/4$.

Another important concept is the notion of distance. Under mild assumptions, this will turn X into a geodesic space. This is another reason why one should not expect the result to holds beyond locally compact, second-countable, Hausdorff spaces.

Definition 5.1.7 *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strongly local regular Dirichlet form on some*

$L^2(X, d\mu)$ with associated energy measure Γ. For any $x, y \in X$, define

$$\begin{aligned}
\rho(x,y) = \rho_{\mathcal{E}}(x,y) &= \sup\{f(x) - f(y) : f \in \mathcal{D}(\mathcal{E}) \cap C_c(X), d\Gamma(f,f) \leq d\mu\} \\
\rho^*(x,y) = \rho^*_{\mathcal{E}}(x,y) &= \sup\{f(x) - f(y) : f \in \mathcal{D}(\mathcal{E}) \cap C(X),\ d\Gamma(f,f) \leq d\mu\}
\end{aligned}$$

Here $d\Gamma(f, f) \leq d\mu$ means $d\Gamma(f, f)$ is absolutely continuous with respect to $d\mu$ and that its Radon-Nykodim derivative bounded by 1 μ-a.e. on X.

The functions ρ, ρ^* depend on both the Dirichlet form and the topology on X. It is lower-semicontinuous, symmetric, and satisfies triangle inequality. It is, however, only pseudo-distance. Moreover, it is possible that $\rho \neq \rho^*$(see [20, 44]).

Definition 5.1.8 *A strongly local regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on some $L^2(X, d\mu)$ is said to satisfy the basic assumption if the following conditions hold*

(a) *The pseudo-distance ρ is actually a distance function on X and X is complete under ρ,*

(b) *The topology induced by ρ is the original topology of X.*

Under these conditions , one also have $\rho = \rho^*$. Moreover, (X, ρ) is a geodesic space and the distance function $f(x) = \rho(x, V)$, where $\emptyset \neq V \subset X$, is in $\mathcal{D}_{loc}(\mathcal{E})$ and that $d\Gamma(f, f) \leq d\mu$[20, 44].

5.2 Poincaré Inequality and Heat Kernel Estimates

Let begins this section with the definition of (weak) Poincaré Inequality.

Definition 5.2.1 *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strongly local regular Dirichlet form on some $L^2(X, d\mu)$ with associated energy measure Γ satisfying the basic assumption. $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is said to satisfy weak Poincaré inequality if there exists constants $k \geq 1$ and $C_P > 0$ such that for any $r > 0, x \in X$ and $f \in \mathcal{D}(\mathcal{E})$,*

$$\min_{\xi \in \mathbb{R}} \int_{B(x,r)} |f - \xi|^2 d\mu \leq C_P r^2 \int_{B(x,kr)} d\Gamma(f, f)$$

If $k = 1$, then $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is said to satisfy Poincaré inequality.

Note that under doubling property, the weak Poincaré inequality and Poincaré inequality are equivalent[39]. Furthermore, doubling property and Poincaré inequality together imply stronger conditions, parabolic Harnack inequality and heat kernel estimates. Recall here that the heat kernel associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a function $p : (0, \infty) \times X \times X \to (0, \infty)$ such that $P_t f(x) = \int f p(t, x, \cdot) d\mu$ for all $f \in L^2(X, d\mu)$ where $P_t, t > 0$ is the heat semigroup associated to $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Definition 5.2.2 *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strongly local regular Dirichlet form on some $L^2(X, d\mu)$ with associated energy measure Γ satisfying the basic assumption. It is said to satisfy the heat kernel estimates if the heat kernel p associates to it satisfies the estimate*

$$\frac{c_1 e^{-\frac{\rho(x,y)^2}{c_2 t}}}{\sqrt{\mu(B(x, \sqrt{t})B(y, \sqrt{t})}} \leq p(t, x, y) \leq \frac{c_3 e^{-\frac{\rho(x,y)^2}{c_4 t}}}{\sqrt{\mu(B(x, \sqrt{t})B(y, \sqrt{t})}}$$

uniformly in $t > 0$ and $x, y \in X$. Here $c_1, c_2, c_3, c_4 > 0$ are fixed constants.

Note that the term $\sqrt{\mu(B(x, \sqrt{t})B(y, \sqrt{t})}$ in the buttom can be replaced either by $\mu(B(x, \sqrt{t}))$ or $\mu(B(y, \sqrt{t}))$[41].

As for parabolic Harnack inequality, one need to clarify first what a solution of heat equation is. This is nothing but a generalization of the classical heat equation. Let L be the infinitesimal generator associates to $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Informally, a solution of the heat equation associated to L is a function u such that $\partial_t u = Lu$. This can also be interpreted as $\int < \partial_t u, v > dt = \int < Lu, v > dt = -\int \mathcal{E}(u, v) dt$ for all test functions v. To make this precise, one must first define test functions.

Given a Hilbert space H and an open interval I, denote $L^2(I \to H)$ the Hilbert space of all measurable function $u : I \to H$ with finite norm

$$||u|| = \Big(\int_I ||u(t)||^2 dt \Big)^{1/2} < \infty$$

Let $W^1(I \to H)$ be the set of all functions $u \in L^2(I \to H)$ whose distributional derivative u' can be represented by a function in $L^2(I \to H)$. Equipped $W^1(I \to H)$ with the norm

$$||u|| = \Big(\int_I ||u(t)||^2 dt + \int_I ||u'(t)||^2 dt \Big)^{1/2}$$

will make $W^1(I \to H)$ into a Hilbert space.

From now on, any function $u : I \to X \to \mathbb{R}$ will be viewed as a function $u : I \to (X \to \mathbb{R})$. This will allow us to view solutions of heat equation as function from $I \to \mathcal{D}(\mathcal{E})$.

Set $\mathcal{F}(I \times X) = L^2(I \to \mathcal{D}(\mathcal{E})) \cap W^1(I \to \mathcal{D}(\mathcal{E})^*)$ and set

$$\mathcal{F}_c(I \times X) = \{u \in \mathcal{F}(I \times X) : u(t, \cdot) \text{ has compact support for a.e. } t \in I\}$$

Also denotes $\mathcal{F}_{loc}(I \times X)$ the set of all functions $u : I \times X \to \mathbb{R}$ such that for any relatively compact open subset V of X and J of I, there exists a function $u_V \in \mathcal{F}(I \times X)$ satisfying $u = u_V$ a.e. on $J \times V$.

Definition 5.2.3 *Let I be an open time interval. A function $u : I \times X \to \mathbb{R}$ is a (local) weak solution of the heat equation $\partial_t u = Lu$ if*

(*a*) $u \in \mathcal{F}_{loc}(I \times X)$,

(*b*) *For any open interval J relatively compact in I and any $\phi \in \mathcal{F}_c(I \times X)$,*

$$\int_J \int_X \phi \partial_t u d\mu dt + \int_J \mathcal{E}(\phi(t,\cdot), u(t,\cdot))dt = 0$$

A simple example of weak solution in the sense introduced above is $u(t,\cdot) = P_t f$ for $t \in I \subset (0,\infty)$, where I is a bounded interval and $f \in L^2(X, d\mu)$. For a more interesting example, one can take a look at Aronson[4] or Gyrya's thesis[17](see also [16]).

Fixed an open set V. Note that if one replace $\mathcal{D}(\mathcal{E})$ with the closure of the set $\{f \in \mathcal{D}(\mathcal{E}) : f$ has compact support in $V\}$ in the above definition, then one also arrives at the definition of local solution (in V) of heat equation[17].

Now, it is possible to define (uniform) parabolic Harnack inequality.

Definition 5.2.4 *A regular strongly local Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(X,\mu)$ satisfies* (uniform) parabolic Harnack inequality *if there exists a constant $H_0 > 0$ such that for any $x \in X$, $r > 0$, and any non-negative weak solution u of the heat equation $\partial_t u = Lu$ on $(0, r^2) \times B(x,r)$, one have*

$$\sup_{Q_-} u \le H_0 \inf_{Q_+} u$$

where $Q_- = (r^2/4, r^2/2) \times B(x, r/2)$, $Q_+ = (3r^2/4, r^2) \times B(x, r/2)$ and both supremum and infimum are essential i.e. computed up to sets of measure zero.

A crucial consequence of uniform parabolic Harnack inequality is that all local weak solutions of the heat equation are continuous in the sense that they admit continuous representatives. Another one is that it is equivalent to heat kernel estimates discuss earlier. The following result is the cornerstone of the author's approach in proving uniform parabolic Harnack inequality. It is the work of Sturm which in turn generalizes the works of many others that came before[47].

Theorem 5.2.5 ([47]) *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a regular strongly local Dirichlet form on $L^2(X, \mu)$ satisfying the basic assumptions. Then the following properties are equivalent:*

(a) $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *satisfies uniform parabolic Harnack inequality.*

(b) $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *satisfies heat kernel estimates.*

(c) $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *satisfies Poincaré inequality and μ satisfies volume doubling property.*

CHAPTER 6

WEIGHTED DIRICHLET SPACES

This chapter is divided into two parts. The first part deals with the construction of weighted Dirichlet spaces. The second part gives sufficient conditions for the heat kernel estimates on weighted Dirichlet spaces. This is equivalent to doubling property and Poincaré inequality[47]. Since doubling property on weighted measures is already study in Chapter 4, this chapter will only deals with the proof of Poincaré inequality.

6.1 Construction of Weighted Dirichlet Spaces

Fixed a locally compact metrisable space X, a radon measure ν on X, and a strongly local, regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ with associated energy measure Γ on $L^2(X, \nu)$. Moreover, we assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ satisfies the basic assumption i.e. the intrinsic metric

$$\rho(x, y) := \sup\{u(x) - u(y) : u \in \mathcal{D}_{loc}(\mathcal{E}) \cap C_c(X),\ d\Gamma(u) \leq d\nu\}$$

is a complete metric metrises the topology of X.

For any $\mathcal{F} \subset \mathcal{D}(\mathcal{E})$, denote $\overline{\mathcal{F}}^{\mathcal{E}}$ the closure of $\mathcal{F}$ under the Dirichlet inner product $\mathcal{E}_1 = \langle\cdot, \cdot\rangle + \mathcal{E}$.

Lemma 6.1.1 *Let $\phi \in C(X)$, $\phi \geq 0$, $u \in \mathcal{D}(\mathcal{E})$, and $v = (u \vee 0) \wedge 1$. Then*

$$\int_X \phi d\Gamma(v, v) \leq \int_X \phi d\Gamma(u, u)$$

Proof. By [6, p.17], the result holds if we further assume that $\phi \in \mathcal{D}(\mathcal{E}) \cap C_c(X)$ and $u \in \mathcal{D}(\mathcal{E}) \cap L^\infty(X, \nu)$. For general u, we have $\int \phi d\Gamma(u) = \sup_n \int \phi d\Gamma((-n) \vee u \wedge n)$, so the result also holds in this case. By regularity of Dirichlet forms, the results can be extended to any $\phi \in C_c(X)$. Since X is a length space, we can extend the result for any $\phi \in C(X)$.

$\square$

Lets starts by assume that the weighted function is regular i.e. it only take finite values.

Lemma 6.1.2 *Let $h : X \to (0, \infty)$ be a continuous function and Ω be a relatively compact, open subset of X. Denote $d\mu := hd\nu$ and*

$$\mathcal{E}^h(u, v) = \int_X hd\Gamma(u, v), \ \forall u, v \in \mathcal{D}(\mathcal{E}) \cap C_c(X)$$

Then $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}) \cap C_c(\Omega))$ is a densely defined, closable, symmetric bilinear form on $L^2(\Omega, \mu)$ and its closure is a strongly local, regular Dirichlet form on $L^2(\Omega, \mu)$ with domain $\overline{\mathcal{D}(\mathcal{E}) \cap C_c(\Omega)}^{\mathcal{E}}$.

Proof. Clearly, $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}) \cap C_c(\Omega))$ is a symmetric bilinear form on $L^2(\Omega, \mu)$.

First, we show that $\mathcal{D}(\mathcal{E}) \cap C_c(\Omega)$ is dense in $L^2(\Omega, \mu)$. Since $C_c(\Omega)$ is dense in $L^2(\Omega, \mu)$, it is sufficient to show that $\mathcal{D}(\mathcal{E}) \cap C_c(\Omega)$ in dense in $C_c(\Omega)$ under supremum norm. Fixed $u \in C_c(\Omega)$. There exist $u_k \in \mathcal{D}(\mathcal{E}) \cap C_c(X)$ such that $\|u_k - u\|_\infty \to 0$ as $k \to \infty$ by regularity of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Choose $F \in \mathcal{D}(\mathcal{E}) \cap C_c(X)$ such that $\chi_{\mathrm{supp}(u)} \le F \le \chi_\Omega$. Such F exists because supp(u) is a compact subset of Ω. Let $v_k = Fu_k \in \mathcal{D}(\mathcal{E}) \cap C_c(\Omega)$. Using $u = Fu$, we have $\|v_k - u\|_\infty \le \|F\|_\infty \|u_k - u\|_\infty \to 0$.

Next, we show that $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}) \cap C_c(\Omega))$ is closable. First notice that $(\mathcal{E}, \mathcal{D}(\mathcal{E}) \cap C_c(\Omega))$ is closable. Denote $m = \inf_{x \in \Omega} h$ and $M = \sup_{x \in \Omega} h$. Since h is continuous and Ω is relatively compact, $0 < m \leq M < \infty$. Now, $md\nu \leq d\mu \leq Md\nu$, and $m\mathcal{E} \leq \mathcal{E}^h \leq M\mathcal{E}$. Hence $\mathcal{E}_1$ and $\mathcal{E}^h_1$ are equivalent. It follows that $(\mathcal{E}^h, \mathcal{D}(\mathcal{E})\cap C_c(\Omega))$ is also closable and its closure has domain $\overline{\mathcal{D}(\mathcal{E})\cap C_c(\Omega)}^{\mathcal{E}}$.

By Theorem 6.1.1, $(\mathcal{E}^h, \overline{\mathcal{D}(\mathcal{E})\cap C_c(\Omega)}^{\mathcal{E}})$ is a Dirichlet form on $L^2(\Omega, \nu)$. Let $u, v \in \overline{\mathcal{D}(\mathcal{E})\cap C_c(\Omega)}^{\mathcal{E}}$ be such that u is constant in a neighborhood of a support of v. Then $d\Gamma(u, v)$ is the zero measure and hence $\mathcal{E}^h(u, v) = 0$. This proof the strong locality of $(\mathcal{E}^h, \overline{\mathcal{D}(\mathcal{E})\cap C_c(\Omega)}^{\mathcal{E}})$.

$\square$

Theorem 6.1.3 *Let $h : X \to (0, \infty)$ be a continuous function. Denote $d\mu := hd\nu$ and*

$$\mathcal{E}^h(u, v) = \int_X hd\Gamma(u, v),\ \forall u, v \in \mathcal{D}(\mathcal{E})\cap C_c(X)$$

Then $(\mathcal{E}^h, \mathcal{D}(\mathcal{E})\cap C_c(X))$ is closable and its closure is a strongly local, regular Dirichlet form satisfies the basic assumption.

Proof. Fixed $o \in X$ and denote $\Omega_n = B(o, n)$. Since X is a length space, Ω_n is relatively compact for all $n \geq 1$. Let $(A_n, \mathcal{D}(A_n))$ be the self-adjoint operator associated to $(\mathcal{E}^h, \overline{\mathcal{D}(E)\cap C_c(\Omega_n)}^{\mathcal{E}})$. Define $Au = A_n u$ for any $u \in \mathcal{D}(E)\cap C_c(\Omega_n)$. Since for smallest possible n, $u = 0$ on a neighborhood of $\Omega_{n+1} - \overline{\Omega_n}$, A is well-defined on $\cup_{n\in\mathbb{N}}\mathcal{D}(E)\cap C_c(\Omega_n) = \mathcal{D}(E)\cap C_c(X)$. It is not hard to see that $(A, \mathcal{D}(E)\cap C_c(X))$ is a densely defined, positive, symmetric operator on $L^2(X, \mu)$. By Friedrichs Extension Theorem, $(\mathcal{E}^h, \mathcal{D}(\mathcal{E})\cap C_c(X))$ is closable. By Theorem 6.1.1, its closure is a regular Dirichlet form on $L^2(X, \mu)$.

Lastly, it is easy to see that the associated energy measure Γ^h of $(\mathcal{E}, \mathcal{D}(\mathcal{E}^h))$ is defined by $d\Gamma^h(u, v) = hd\Gamma(u, v)$ and hence $d\Gamma^h(u) \leq d\mu$ if and only if $d\Gamma(u) \leq d\nu$.

Thus, $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ also satisfies the basic assumption. Moreover, it defines the same metric as $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

$\square$

Theorem 6.1.4 (Friedrichs Extension) *Let A be a positive symmetric densely defined linear operator with domain $\mathcal{D}(A)$ in a Hilbert space and let $q(x, y) := \langle Ax, y\rangle$ for any $x, y \in \mathcal{D}(A)$. Then q is a closable symmetric bilinear form.*

Recall that a densely defined linear operator A is positive if $\langle Ax, x\rangle \geq 0$ for any $x \in \mathcal{D}(A)$, A is symmetric if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for any $x, y \in \mathcal{D}(A)$.

Proof. See [33, p.195,255] and [34, p.177].

$\square$

Now lets move on to singular weights. Let $h : X \to (0, \infty]$ be a positive continuous, locally integrable function on X and $d\mu = hd\nu$. Here h is continuous means h is lower semi-continuous on X and continuous on $X - \{h = \infty\}$ i.e.

$$h(x_n) \to h(x) \quad \text{whenever} \quad x_n \to x \in \{h \neq \infty\}$$
$$h(x_n) \to \infty \quad \text{whenever} \quad x_n \to x \in \{h = \infty\}$$

Note that we can always write $h = h_1 h_2$ where h_1 is bounded from above and h_2 is bounded from below. One such choices is that $h_1 = h \wedge 1$ and $h_2 = h \vee 1$.

Define $\mathcal{E}'(u, v) = \int h_1 d\Gamma(u, v)$ for any $u, v \in \mathcal{D}(\mathcal{E})$. The previous subsection show that $(\mathcal{E}', \mathcal{D}(\mathcal{E}) \cap C_c(X))$ is closable and its closure $(\mathcal{E}', \mathcal{D}(\mathcal{E}'))$ is a strongly local, regular Dirichlet form satisfies the basic assumption. One can then replace

$(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ with $(\mathcal{E}', \mathcal{D}(\mathcal{E}'))$ and assume the following stronger assumptions for the weight h.

Assumption 6.1.1 *$h : X \to (0, \infty]$ is continuous, locally integrable and has positive minimum 1. Particularly, $\nu(h = \infty) = 0$.*

Theorem 6.1.5 *Let $h : X \to (0, \infty]$ be a continuous, locally integrable function with positive minimum 1. Denote $d\mu := hd\nu$ and*

$$\mathcal{E}^h(u, v) = \int_X hd\Gamma(u, v),\ \forall u, v \in \mathcal{D}(\mathcal{E}) \cap C_c(X)$$

If $C = \{u \in \mathcal{D}(\mathcal{E}) \cap C_c(X) : \mathcal{E}^h(u) < \infty\}$ is dense in $(C_c(X), \|\cdot\|_\infty)$, then $(\mathcal{E}^h, C)$ is closable and its closure is a strongly local, regular Dirichlet form on $L^2(X, hd\nu)$ satisfies the basic assumption.

Proof. The assumption about C is to guarantee that $(\mathcal{E}^h, C)$ is at least densely defined and its closure, once proved, is regular. Let (u_n) be a Cauchy sequence in $(\mathcal{E}^h, C)$ such that $u_n \to 0$ in $L^2(X, hd\nu)$. Since $h \geq 1$, $u_n \to 0$ in $L^2(X, \nu)$ and $\mathcal{E}(u_m - u_n) \to 0$ as $n, m \to \infty$. It follows that $\mathcal{E}(u_n) \to 0$.

Using the fact that $|\int |\phi| d\Gamma(u_n, u_m)| \leq \sqrt{\int |\phi| d\Gamma(u_n, u_n) \int |\phi| d\Gamma(u_m, u_m)}$ and $\int |\phi| d\Gamma(u_n) \leq \|\phi\|_\infty \mathcal{E}(u_n)$ for all n, m, we have

$$\lim_{m\to\infty} \int |\phi| d\Gamma(u_m, u_m) = \lim_{m\to\infty} \int |\phi| d\Gamma(u_n, u_m) = 0,\ \forall \phi \in C(X) \cap L^\infty(X, \nu)$$

Set $U_k = \{k-1 < h < k+1\}$. Since $X - \{h = \infty\}$ is a metric space, it is paracompact and hence there exists a partition of unity $\{\phi_k\}$ subordinates to $\{U_k\}$. Then

$$\int_X hd\Gamma(u_n) \quad \leq \quad \sum_{k\in\mathbb{N}} \int_X \phi_k hd\Gamma(u_n)$$

$$
\begin{aligned}
&= \sum_{k\in\mathbb{N}} \lim_{m\to\infty} \int_X \phi_k h d\Gamma(u_n - u_m) \\
&\leq \liminf_{m\to\infty} \sum_{k\in\mathbb{N}} \int_X \phi_k h d\Gamma(u_n - u_m) \\
&\leq \liminf_{m\to\infty} \int_X h d\Gamma(u_n - u_m)
\end{aligned}
$$

Letting $n \to \infty$ and we have $\mathcal{E}^h(u_n) \to 0$. This proves that $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}) \cap C_c(X))$ is closable. Using the fact that $\mathcal{E}_1 \leq \mathcal{E}_1^h$, $\mathcal{D}(\mathcal{E}^h) := \overline{\mathcal{D}(\mathcal{E}) \cap C_c(X)}^{\mathcal{E}^h} \subset \mathcal{D}(\mathcal{E})$.

By Lemma 6.1.1, $\int_X h \wedge k d\Gamma(v) \leq \int_X h \wedge k d\Gamma(u)$ for any $u \in \mathcal{D}(\mathcal{E}^h)$ and $v = (u \vee 0) \wedge 1$. Letting $k \to \infty$ and we have $\mathcal{E}^h(v) \leq \mathcal{E}^h(u)$ which prove that $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ is a Dirichlet form. It is not hard to see that in fact $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ is a strongly local, regular Dirichlet form on $L^2(X, hd\nu)$ satisfies the basic assumption.

$\square$

Corollary 6.1.6 *Let $h : X \to (0, \infty]$ be a continuous, locally integrable function. Denote $d\mu := hd\nu$ and*

$$
\mathcal{E}^h(u, v) = \int_X h d\Gamma(u, v), \ \forall u, v \in \mathcal{D}(\mathcal{E}) \cap C_c(X)
$$

Let $C = \{u \in \mathcal{D}(\mathcal{E}) \cap C_c(X) : \mathcal{E}^h(u) < \infty\}$. If $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ admits a carré du champ operator, then $(\mathcal{E}^h, C)$ is closable and its closure is a strongly local, regular Dirichlet form on $L^2(X, hd\nu)$ satisfies the basic assumption.

Proof. This follows from the fact that $\mathcal{E}^h(u) < \infty$ for any Lipschitz function with compact support.

$\square$

Denote $S = S_h := \{h = \infty\}$ where $h : X \to (0, \infty]$ be a continuous, locally integrable function. Instead of construct the weighted Dirichlet form as the closure of $C = \{u \in \mathcal{D}(\mathcal{E}) \cap C_c(X) : \mathcal{E}^h(u) < \infty\}$, one might replace it with $\mathcal{D}(\mathcal{E}) \cap C_c(X - S)$. The last goal of this section is to give a sufficient condition for which there is no different between the two i.e. the sufficient condition for which $\mathcal{D}(\mathcal{E}^h) = \overline{\mathcal{D}(\mathcal{E} \cap C_c(X-S))}^{\mathcal{E}^h}$.

Theorem 6.1.7 *Assume that the weighted Dirichlet form $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ is well-defined, and for each compact subset K of $S = \{h = \infty\}$,*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{K^\varepsilon} h d\nu = 0 \tag{6.1}$$

Then $\mathcal{D}(\mathcal{E}^h) = \overline{\mathcal{D}(\mathcal{E}) \cap C_c(X-S)}^{\mathcal{E}^h}$.

The proof is based on the following lemma.

Lemma 6.1.8 *Let $f : [0, \infty) \to [0, \infty)$ be a nondecreasing function with $f(0) = 0$. Then*

$$\exists \varepsilon_n \searrow 0 \; s.t. \; f(\varepsilon_n)/\varepsilon_n^2 \to 0 \iff \exists \varepsilon_n \searrow 0 \; s.t. \; f(\varepsilon_n)/(\varepsilon_n - \varepsilon_{n+1})^2 \to 0$$

Proof. ($\Rightarrow$) Choose a subsequence ε_{n_k} so that $\varepsilon_{n_{k+1}}/\varepsilon_{n_k} \to 0$. Then

$$\begin{aligned} \frac{f(\varepsilon_{n_k})}{(\varepsilon_{n_k} - \varepsilon_{n_{k+1}})^2} &= \frac{f(\varepsilon_{n_k})}{\varepsilon_{n_k}^2} \frac{\varepsilon_{n_k}^2}{(\varepsilon_{n_k} - \varepsilon_{n_{k+1}})^2} \\ &\to 0 \end{aligned}$$

($\Leftarrow$) Since $\varepsilon_n \geq \varepsilon_n - \varepsilon_{n+1}$, $f(\varepsilon_n)/\varepsilon_n^2 \leq f(\varepsilon_n)/(\varepsilon_n - \varepsilon_{n+1})^2 \to 0$.

$\square$

Theorem 6.1.7. For any $\varepsilon > 0$ and $K \subset X$, denote K^ε the ε-neighborhood of K. It is sufficient to show that $\mathcal{D}(\mathcal{E}) \cap C_c(X) \cap \{u : \mathcal{E}^h(u) < \infty\} \subset \overline{\mathcal{D}(\mathcal{E}) \cap C_c(X - S)}^{\mathcal{E}^h}$. Let $u \in \mathcal{D}(\mathcal{E}) \cap C_c(X)$ be such that $\mathcal{E}^h(u) < \infty$ and set $K = \{h = \infty\} \cap \mathrm{supp}(u)$, $f(\varepsilon) = \int_{K^\varepsilon} h d\nu$. By assumption, we can choose $\varepsilon_n \searrow 0$ so that $f(\varepsilon_n)/(\varepsilon_n - \varepsilon_{n+1})^2 \to 0$. Set

$$\rho_n(x) = \begin{cases} \frac{\rho(x, K^{\varepsilon_{n+1}})}{\varepsilon_n - \varepsilon_{n+1}} & x \in K^{\varepsilon_n} \\ 1 & \text{otherwise} \end{cases}$$

Then $u_n = \rho_n u \in \mathcal{D}(\mathcal{E}) \cap C_c(X - S)$, $u_n \to u$ a.e. and hence

$$\int (u_n - u)^2 d\nu \leq (\sup |u|)^2 \int_{K^{\varepsilon_n}} d\nu \to 0$$

Since $\rho_n - 1$ is Lipschitz with constant $2/(\varepsilon_n - \varepsilon_{n+1})$,

$$\begin{aligned} \int h d\Gamma(u_n - u, u_n - u) &\leq 2 \int [u^2 h d\Gamma(\rho_n - 1, \rho_n - 1) + (\rho_n - 1)^2 h d\Gamma(u, u)] \\ &\leq 2 \sup |u|^2 f(\varepsilon_n)/(\varepsilon_n - \varepsilon_{n+1})^2 + 2 \int_{K^{\varepsilon_n}} h d\Gamma(u, u) \\ &\to 0 \end{aligned}$$

Therefore, $\|u_n - u\|_{\mathcal{E}^h} \to 0$.

□

The above theorem shows that if the dimension of S is too small compared to the growth rate of the measure, then it does not matter which domain one should to prove the Poincaré inequality. This idea is not exploited in this thesis though.

6.2 The proof of Poincaré inequality

As state in the background materials, heat kernel estimates is equivalent to doubling property and Poincaré inequality. The doubling property is studied in earlier chapter, so in this section, the author will focus on Poincaré inequality. The first version of the result will focus on nonincreasing remotely constant weights, while the second one will focus on measure with small growth. Unlike doubling property with work well with increasing weights, it is simpler to prove Poincaré inequality for nonincreasing weights.

Another important point is that the proof of Poincaré inequality relies on paths between points, so the singularity sets must have some kinds of path property too.

Definition 6.2.1 *Let (X, d) be a geodesic space and $\Sigma \subset X$. The set Σ is said to be ρ-accessible if it satisfies ρ'-skew condition for some $\rho' > \rho$ and the cone $c\Sigma_\rho(o, r)$ is path-connected for all $o \in \Sigma$ and $r > 0$.*

A ρ-accessible set Σ is said to be ρ-couniform if the $\Sigma_\rho(o, r)$ itself is path-connected for all $o \in \Sigma$ and $r > 0$.

Although the definition of accessible set does not involve the length of the path, it is still possible to control it.

Proposition 6.2.2 *Let Σ be a ρ-accessible set in a geodesic space (X, d). For each $\rho_0 < \rho$, there exists a constant $C_L > 0$ such that for any $x \in c\Sigma_\rho(o)$, one can find a path $\gamma \subset c\Sigma_{\rho_0}(o)$ from o to x with length at most $C_L d(o, x)$.*

Proof. First, choose $\varepsilon > 0$ so that $\rho > (1+2\varepsilon)\rho_0 + 2\varepsilon$. For any $x \in c\Sigma_\rho(o, r)$, one can find a path $\gamma \subset c\Sigma_\rho(o, r)$ connecting x to o. For each $k = 1, 2, \ldots$, denote t_k the last point that $d(o, \gamma(t)) \geq \rho^k r$, and also denote $t_0 = 0$. On each interval $[t_k, t_{k+1}]$, replace the path with the path constructed below.

Denote $\{x_i\}$ an $\varepsilon\rho^k r$-set on the path $\gamma|_{[t_k, t_{k+1}]}$. By doubling property, the number of $\{x_i\}$ is uniformly bounded depending only on doubling constant and ε. Rearranging $\{x_i\}$ so that $B(x_i, 2\varepsilon\rho^k r) \cap B(x_{i+1}, 2\varepsilon\rho^k r) \neq \emptyset$. Now, one can replace $\gamma|_{[t_k, t_{k+1}]}$ with the path in this chain. It is easy to see that this new path has length roughly $\rho^k r$ on $[t_k, t_{k+1}]$ so the whole path has length roughly r.

Lastly, for any $y \in B(x_i, \varepsilon\rho^k r)$,

$$d(y, o) \leq d(y, x_i) + d(x_i, o) \leq (1+\varepsilon)\rho^k r$$

and

$$d(y, \Sigma) \geq d(x_i, \Sigma) - d(x_i, y) \geq (\rho - \varepsilon)\rho^k r$$

Therefore, $y \in c\Sigma_{\rho_0}(o)$.

□

By the same arguments, one can also prove the following result.

Proposition 6.2.3 *Let Σ be a ρ-couniform set in a geodesic space (X, d). For each $\rho_0 < \rho$, there exists a constant $C_L > 0$ such that for any $x, y \in \Sigma_\rho(o, r)$, one can find a path $\gamma \subset \Sigma_{\rho_0}(o, r)$ from x to y with length at most $C_L d(x, y)$.*

The next result gives the reason why one should not simply assume Σ satisfies ρ-skew condition.

Lemma 6.2.4 *Let $x \in c\Sigma_\rho(o)$, $o \in \Sigma$. Then for each $\epsilon \in (0,1)$ and $\tilde{o} \in \Sigma$ with $d(\tilde{o}, o) \leq \epsilon d(x, \Sigma)$, one have $x \in c\Sigma_{\frac{\rho}{1+\rho\epsilon}}(\tilde{o})$.*

Proof. Let $x \in \Sigma_\rho(o)$, and $d(\tilde{o}, o) \leq \epsilon d(x, \Sigma)$.

$$\begin{aligned} d(x, \tilde{o}) &\leq d(x, o) + d(o, \tilde{o}) \\ &\leq (\rho^{-1} + \epsilon) d(x, \Sigma) \end{aligned}$$

Therefore $d(x, \Sigma) \geq \frac{\rho}{1+\rho\epsilon} d(x, \tilde{o})$.

□

Proposition 6.2.5 *Let Σ be a subset of a geodesic space (X, d) satisfying ρ-skew condition and $2\rho < \rho'$. Then for any $r > 0$ and $o, o' \in \Sigma$ with $d(o, o') \leq r$,*

$$\Sigma_{\rho'}(o, r) \subset c\Sigma_\rho(o', 2r) - B(o', 2\rho r)$$

Proof. Clearly, for any $x \in \Sigma_{\rho'}(o, r)$, $d(x, o') \leq d(x, o) + d(o, o') \leq 2r$.

□

Now, lets prove Poincaré inequality. There are three versions of the theorem. The first one is when Σ is couniform. This case is not new; it is proved in Saloff-Coste and Gyrya[17] result. The proof given here is an adaptation of Saloff-Coste and Gyrya's proof. It is modified to give a basic building block of the other cases. The second version deals with nonincreasing weight functions when Σ is only require to be accessible. The last version try to generalize the result to a bigger class of function by assume some dimensional conditions on Σ.

All versions are based on the following theorem.

Theorem 6.2.6 *Let $(\mathcal{E}, \mathcal{D}(E))$ be a strongly local, regular, Dirichlet form in $L^2(X, \mu)$ with intrinsic metric d satisfies the usual assumptions. Let also $\Sigma \subset X$ be a ρ-accessible subset. Assume that the measure μ is doubling and the local Poincare inequality holds for all Σ-remote balls. Then the local Poincare inequality holds for all balls if and only if there exists a constant $C_A > 0$ such that for any $o \in \Sigma$, $x, y \in \Sigma_\rho(o, r)$, and $f \in \mathcal{D}(\mathcal{E})$,*

$$\fint_{B(x,\varepsilon r)} \fint_{B(y,\varepsilon r)} |f(u) - f(v)|^2 d\mu(u) d\mu(v) \leq C_A r^2 \fint_{B(o,r)} d\Gamma(f, f) \tag{6.2}$$

where $0 < \varepsilon < 1$ be a fixed constant independent of o, x, y, r and f.

The proof relies on the Whitney covering so lets us review the definition here. A strict ϵ-Whitney covering of $X - \Sigma$, denoted $\mathcal{W}$, is a countable family of disjoint balls $B(x, r)$ such that $r = \epsilon d(x, \Sigma)$ and $\cup_{A \in \mathcal{W}} 3A = X - \Sigma$. Here ϵ is a very small positive number. For any $o \in \Sigma$, denote $\mathcal{W}_o = \{A \in \mathcal{W} : 3A \cap c\Sigma_\rho(o) \neq \emptyset\}$.

In view of Saloff-Coste and Grigor'yan's work[10], it is sufficient to prove the result for anchored balls. The arguments relies on the chain arguments and is divided into 4 steps. First one connects any remote balls to another remote balls of roughly the same radius and can be connected to a balls of radius roughly the same size of the anchored ball. Then connects that ball to a fixed remote ball of radius roughly that of the anchored balls. The difficulty lies in controling the number of times a ball is used.

Lemma 6.2.7 *Let Σ be a ρ-accessible set in a geodesic space (X, d). Fixed $\varepsilon \in (0, 1]$. There exists a constant $N > 1$ independent of $o \in \Sigma$ and $r > 0$ such that the number of $A \in \mathcal{W}_o$ with $\varepsilon r \leq r(A) \leq r$ is at most N.*

Proof. For such $A = B(x, s)$, denote $y \in c\Sigma_\rho(o) \cap A$. Note that

$$d(x, o) \leq d(x, y) + d(y, o)$$

$$
\begin{aligned}
&\leq s+\rho^{-1}d(y,\Sigma)\\
&\leq s+\rho^{-1}(d(y,x)+d(x,\Sigma))\\
&\leq s+\rho^{-1}(s+\epsilon^{-1}s)\\
&= (1+\rho^{-1}+\rho^{-1}\epsilon^{-1})s
\end{aligned}
$$

Therefore $A \subset B(o,(2+\rho^{-1}+\rho^{-1}\epsilon^{-1})r)$. By doubling property, there is a number N depend only on doubling constant, ρ, ϵ, and ε such that the number of such A is always at most N.

□

Lemma 6.2.8 *Let Σ be a ρ-accessible set in a geodesic space (X,d). There exists a constant $N, C \geq 1$ such that for any $x \in c\Sigma_\rho(o), o \in \Sigma$, we can find a sequence of remote balls $\{B_i = B(x_i, r_i)\}_{i=1}^{\infty} \subset \mathcal{W}$ with the following properties*

1. $x \in 3B_1$,
2. $3B_i \cap 3B_j \neq \emptyset$ *if and only if* $|i-j| = 1$,
3. $\frac{1}{C}\epsilon^i d(x,o) \leq r_i \leq C\epsilon^{i/N} d(x,o)$,
4. *there exists a partition* $\mathcal{P}_k$ *of* $\{B_i\}$, *each with at most* N *elements such that* $\frac{1}{C}\epsilon^k d(x,o) \leq r_i \leq C\epsilon^k d(x,o)$ *for any* $B_i \in \mathcal{P}_k$
5. $B_j \subset CB_i$ *for any* $j > i$

Proof. Fixed $\rho_0 < \rho$. Denote $d(x,o) = r$. By assumptions, there is a path γ from x to o lying entirely in $c\Sigma_{\rho_0}(o,r)$ such that there is a unique t_i with $d(\gamma(t_i),o) = \epsilon^i r$ and $\gamma([t_i,t_{i+1}]) \subset \Sigma_{\rho_0}(o,r)$. The path γ can be constructed recursively on each level.

For each $A \in \mathcal{W}$ that intersects $\gamma([t_i, t_{i+1}])$, $\frac{\epsilon^{i+1}}{1+\epsilon} r \leq r(A) \leq \frac{\epsilon^i}{1-\epsilon} r$. Therefore there exists $N \geq 1$ such that at most N such A cover $\gamma([t_i, t_{i+1}])$.

Choose $B_1 \in \mathcal{W}$ so that $x \in 3B_1$ then choose B_i recursively as follows. Let s_i be the last s so that $d(\gamma(s), 3B_i) = 0$. If $s_i \leq t_1$ we are done for this level, otherwise choose $B_{i+1} \in \mathcal{W}$ that contains $\gamma(s_i)$. This process will be done in at most N steps. After this repeat the process on each level recursively. It is clear that this sequence satisfies the first four properties. For the last one, the distance from x_i to x_j is at most

$$N \sum_{k=1}^{j-i} \frac{(1+\epsilon)\epsilon^k r_i}{1-\epsilon} \leq \frac{N(1+\epsilon)}{(1-\epsilon)^2} r_i$$

$\square$

Now, one can prove the above theorem.

Proof of Theorem 6.2.6. First choose $\rho' > \rho$ such that Σ satisfies ρ'-skew condition and $0 < \epsilon < 1$ so that $\frac{\rho'}{1+\rho'\epsilon} > \rho$. Fixed $o' \in \Sigma$ and $r > 0$. For each $o \in B(o', 3r) \cap \Sigma$, choose $x_o \in \Sigma_{\rho'}(o, \frac{3r}{\rho\epsilon})$. This is possible by ρ'-skew condition. It follows that $x_o \in c\Sigma_\rho(o')$. Choose, for each x_o, a sequence of remote balls $\mathcal{B}_o = \{B_i^o = B(x_i, r_i)\}_{i=1}^\infty \subset \mathcal{W}$ according to Lemma 6.2.8.

Let $\mathcal{F} = \{B \in \mathcal{W} : 3B \cap B(o', r) \neq \emptyset\}$. For each $B = B(z, r_B) \in \mathcal{F}$, choose $o_B \in \Sigma$ so that $z \in c\Sigma_\rho(o_B)$. Notice that $d(o', o_B) \leq 3r$. Then choose $W_B \in \mathcal{B}_{o_B}$ to be the first ball that has radius at most r_B. Since the radius of W_B is roughly the same as that of the one before it, its radius must be roughly r_B.

Next, denote $W_o = B_1^o$ and $W = B_1^{o'}$. Denote also $f_B = \fint_{4B} f \, d\mu$ for any balls B. Nothing that

$$|f - f_W|^2 \leq 4\Big[|f - f_B|^2 + |f_B - f_{W_B}|^2 + |f_{W_B} - f_{W_{o_B}}|^2 + |f_{W_{o_B}} - f_W|^2\Big]$$

for any $B \in \mathcal{F}$. Therefore

$$\int_{B(o',r)} |f - f_W|^2 d\mu \leq 4 \sum_{B \in \mathcal{F}} \int_{4B} \Big[|f - f_B|^2 + |f_B - f_{W_B}|^2 + |f_{W_B} - f_{W_{o_B}}|^2 + |f_{W_{o_B}} - f_W|^2 \Big] d\mu$$

The first term is bounded easily. For any $B \in \mathcal{F}$, $4B \subset B(o', 2r)$. The Poincaré inequality for remote balls then implies

$$\begin{aligned}
\sum_{B \in \mathcal{F}} \int_{4B} |f - f_B|^2 d\mu &\lesssim r^2 \sum_{B \in \mathcal{F}} \int_{4B} d\Gamma(f, f) \\
&\lesssim r^2 \int_{B(o',2r)} \Big(\sum_{B \in \mathcal{F}} \chi_{4B} \Big) d\Gamma(f, f) \\
&\lesssim r^2 \int_{B(o',2r)} d\Gamma(f, f)
\end{aligned}$$

The last term is also simple to bounded. Since the radius of each W_o is roughly r, there must be at most N different balls A_i by Lemma 6.2.7. Therefore

$$\begin{aligned}
\sum_{B \in \mathcal{F}} \int_{4B} |f_{W_{o_B}} - f_W|^2 d\mu &= \sum_i |f_{A_i} - f_W|^2 \Big(\sum_{W_{o_B} = A_i} \mu(4B) \Big) \\
&\lesssim \sum_i |f_{A_i} - f_W|^2 \mu(B(o', \frac{3r}{\rho\epsilon})) \\
&\lesssim \sum_i \fint_{4A_i} \fint_{4W} |f(u) - f(v)|^2 d\mu(u) d\mu(v) \mu(B(o', \frac{3r}{\rho\epsilon})) \\
&\lesssim N r^2 \int_{B(o', \frac{3r}{\rho\epsilon})} d\Gamma(f, f)
\end{aligned}$$

where the last inequality follows from the assumption.

The second term can also be bounded using the same idea, even though it is a bit more complicated. The map $B \mapsto W_B$ is many to one but can be uniformly bounded using doubling property. Since the radius of B and W_B are roughly the same, any Whitney balls that maps to the same W_B must have roughly the same radius. Moreover, the distance from this ball to W_B is also roughly equal the radius of W_B too. Therefore, the number of B that maps to W_B must be uniformly bounded by doubling property. Therefore,

$$\begin{aligned}\int_{4B} |f_B - f_{W_B}|^2 d\mu &\leq \mu(4B) \fint_{4B} \fint_{4W_B} |f(u) - f(v)|^2 d\mu(u) d\mu(v) \\ &\lesssim r_B^2 \int_{B(o_B, r_B/\epsilon)} d\Gamma(f, f)\end{aligned}$$

Denote $\mathcal{F}_k$ the set of all those balls in $\mathcal{F}$ with radius between $3r/2^k$ and $3r/2^{k+1}$. For any $B, B' \in \mathcal{F}_k$ that $B(o_B, r_B/\epsilon) \cap B(o_{B'}, r_{B'}/\epsilon) \neq \emptyset$, the distance between B and B' must be roughly the same as the radius of B. This implies that $\sum_{B \in \mathcal{F}_k} \chi_{B(o_B, r_B/\epsilon)}$ can be uniformly bounded. Hence,

$$\begin{aligned}\sum_{B \in \mathcal{F}} \int_{4B} |f_B - f_{W_B}|^2 d\mu &= \sum_k \sum_{B \in \mathcal{F}_k} \int_{4B} |f_B - f_{W_B}|^2 d\mu \\ &\lesssim \sum_k \sum_{B \in \mathcal{F}_k} \left(\frac{3r}{2^k}\right)^2 \int_{B(o_B, r_B/\epsilon)} \Gamma(f, f) d\mu \\ &\lesssim \sum_k \left(\frac{3r}{2^k}\right)^2 \int_{B(o', \kappa r)} d\Gamma(f, f) \\ &\lesssim r^2 \int_{B(o', \kappa r)} d\Gamma(f, f)\end{aligned}$$

Lastly, for the third term, we learned that there are only a finite number of different W_{o_B} denoted A_i. Each B also pair up with only a finite number of W_B, therefore the third term is comparable to

$$\sum_i \sum_{A \in \mathcal{A}_i} \int_{4A} |f_A - f_{A_i}|^2 d\mu$$

where $\mathcal{A}_i$ is the set of all W_B such that $W_{o_B} = A_i$.

Denote $\mathcal{R}_i = \cup_{W_{o_B} = A_i} \mathcal{B}_{o_B}$. For each $A \in \mathcal{A}_i$, $A = B_k^o \in \mathcal{B}_o$ for some o and k. Using the fact that $A \subset C B_j^o$ for all $j < k$, we have

$$|f_A - f_{A_i}| \chi_A \leq \sum_{j=1}^{k} |f_{B_j^o} - f_{B_{j-1}^o}| \chi_A$$

$$
\begin{aligned}
&\lesssim \sum_{j=1}^{k} r\Big(\frac{1}{\mu(B_j^o)}\int_{16B_j^o} d\Gamma(f,f)\Big)^{1/2}\chi_A\chi_{CB_j^o} \\
&\lesssim \sum_{R\in\mathcal{R}_i} r\Big(\frac{1}{\mu(R)}\int_{16R} d\Gamma(f,f)\Big)^{1/2}\chi_A\chi_{CR}
\end{aligned}
$$

Here the second inequality follows from the following estimates

$$
\begin{aligned}
\mu(B_j^o)|f_{B_j^o}-f_{B_{j-1}^o}|^2 &\sim \mu(4B_j^o\cap 4B_{j-1}^o)|f_{B_j^o}-f_{B_{j-1}^o}|^2 \\
&= \int_{4B_j^o\cap 4B_{j-1}^o} |f_{B_j^o}-f_{B_{j-1}^o}|^2 d\mu \\
&\lesssim \int_{4B_j^o\cap 4B_{j-1}^o} |f-f_{B_{j-1}^o}|^2 d\mu + \int_{4B_j^o\cap 4B_{j-1}^o} |f_{B_j^o}-f|^2 d\mu \\
&\lesssim \int_{4B_{j-1}^o} |f-f_{B_{j-1}^o}|^2 d\mu + \int_{4B_j^o} |f_{B_j^o}-f|^2 d\mu \\
&\lesssim r^2\int_{4B_{j-1}^o} d\Gamma(f,f) + \int_{4B_j^o} d\Gamma(f,f) \\
&\lesssim r^2\int_{16B_j^o} d\Gamma(f,f)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\sum_{A\in\mathcal{A}_i}\int_{4A}|f_A-f_{A_i}|^2 d\mu &\sim \sum_{A\in\mathcal{A}_i}\int_{A}|f_A-f_{A_i}|^2 d\mu \\
&\lesssim \int \sum_{A\in\mathcal{A}_i}\Big(\sum_{R\in\mathcal{R}_i} r\Big(\frac{1}{\mu(R)}\int_{16R} d\Gamma(f,f)\Big)^{1/2}\chi_A\chi_{CR}\Big)^2 d\mu \\
&\lesssim r^2\int \Big(\sum_{A\in\mathcal{A}_i}\chi_A\Big)\Big(\sum_{R\in\mathcal{R}_i}\Big(\frac{1}{\mu(R)}\int_{16R} d\Gamma(f,f)\Big)^{1/2}\chi_{CR}\Big)^2 d\mu \\
&\lesssim r^2\int \Big(\sum_{R\in\mathcal{R}_i}\Big(\frac{1}{\mu(R)}\int_{16R} d\Gamma(f,f)\Big)^{1/2}\chi_{CR}\Big)^2 d\mu \\
&\lesssim r^2\int \Big(\sum_{R\in\mathcal{R}_i}\Big(\frac{1}{\mu(R)}\int_{16R} d\Gamma(f,f)\Big)^{1/2}\chi_{R}\Big)^2 d\mu
\end{aligned}
$$

where the last inequality follows from Lemma 3.3.5.

Since balls in $\mathcal{R}_i$ are disjoint

$$
\sum_{A\in\mathcal{A}_i}\int_{4A}|f_A-f_{A_i}|^2 d\mu \lesssim r^2\int \sum_{R\in\mathcal{R}_i}\frac{1}{\mu(R)}\int_{16R} d\Gamma(f,f)\chi_R d\mu
$$

$$\lesssim r^2 \int \sum_{R \in \mathcal{R}_i} \chi_{16R} d\Gamma(f, f)$$
$$\lesssim r^2 \int_{B(o', \kappa r)} d\Gamma(f, f)$$

$\square$

Corollary 6.2.9 *Let $(\mathcal{E}, \mathcal{D}(E))$ be a strongly local, regular, Dirichlet form in $L^2(X, \mu)$ with intrinsic metric d satisfies the usual assumptions. Let also $\Sigma \subset X$ be a ρ-couniform subset. If the measure μ is doubling and the local Poincare inequality holds for all Σ-remote balls, then the local Poincare inequality holds for all balls.*

Proof. The proof follows from the chain condition technique and the fact that $\Sigma_\rho(o, r)$ are connected for all $o \in \Sigma$ and $r > 0$.

$\square$

Corollary 6.2.10 *Let $(\mathcal{E}, \mathcal{D}(E))$ be a strongly local, regular, Dirichlet form in $L^2(X, \nu)$ with intrinsic metric d satisfies the usual assumptions. Let also $\Sigma \subset X$ be a ρ-accessible subset, $d\mu = hd\nu$ where $h(x) = a(d(x, \Sigma))$, and $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ be the weighted Dirichlet form corresponding to function h.*

Assume $(\mathcal{E}, \mathcal{D}(E))$ satisfies parabolic Harnack inequality and a is nonincreasing, and remotely constant. Then the local Poincare inequality for $(\mathcal{E}^h, \mathcal{D}(\mathcal{E}^h))$ holds for all balls whenever μ is doubling.

Proof. Since a is remotely constant,

$$⨍_{B(x,\varepsilon r)} ⨍_{B(y,\varepsilon r)} |f(u) - f(v)|^2 d\mu(x) d\mu(y) \sim ⨍_{B(x,\varepsilon r)} ⨍_{B(y,\varepsilon r)} |f(u) - f(v)|^2 d\nu(x) d\nu(y)$$

The fact that a is nonincreasing implies $a(r)\int_{B(o,r)} d\Gamma(f,f) \le \int_{B(o,r)} hd\Gamma(f,f)$. The doubling property $a(r)\nu(B(o,r)) \sim \mu(B(o,r))$ then implies $⨍_{B(o,r)} d\Gamma(f,f) \le ⨍_{B(o,r)} hd\Gamma(f,f)$.

□

Corollary 6.2.11 *Let $(\mathcal{E},\mathcal{D}(E))$ be a strongly local, regular, Dirichlet form in $L^2(X,\mu)$ with intrinsic metric d satisfies the usual assumptions. Let also $\Sigma \subset X$ be a ρ-accessible subset.* Assume that the growth rate of μ on Σ is less than 2. *If the measure μ is doubling and the local Poincare inequality holds for all Σ-remote balls, then the local Poincare inequality holds for all balls.*

Proof. Let $\beta < 2$ denote the growth rate of μ. Fixed $x \in \Sigma_\rho(o,r)$. Let B_i be the sequence of balls defined in Lemma 6.2.8. Then

$$\begin{aligned}
|f_{B_1} - f(o)|^2 &\le \Big(\sum_{i=1}^{\infty} |f_{B_i} - f_{B_{i+1}}|\Big)^2 \\
&\lesssim \Big(\sum_{i=1}^{\infty} \frac{r}{2^i}\Big(\frac{1}{\mu(B_i)}\int_{16B_i} hd\Gamma(f,f)\Big)^{1/2}\Big)^2 \\
&\sim r^2\Big(\sum_{i=1}^{\infty}\Big(\frac{2^{i(\beta-2)}}{\mu(B_1)}\int_{16B_i} hd\Gamma(f,f)\Big)^{1/2}\Big)^2 \\
&\lesssim r^2\frac{1}{\mu(B_1)}\int_{B(o,2r)} hd\Gamma(f,f)(\sum_{i=1}^{\infty} 2^{i(\beta-2)})^2 \\
&\sim r^2\frac{1}{\mu(B_1)}\int_{B(o,2r)} hd\Gamma(f,f)
\end{aligned}$$

This implies $\frac{1}{\mu(B_1)}\int_{4B_1} |f - f(o)|^2 d\mu \le r^2\frac{1}{\mu(B_1)}\int_{B(o,2r)} hd\Gamma(f,f)$ as well.

□

BIBLIOGRAPHY

[1] K.S. Lau A. Grigor'yan, J. Hu. Heat kernels on metric spaces with doubling measure. In *Fractal Geometry and Stochastics IV*, volume 61 of *Progress in Probability*. Springer, 2009.

[2] Adam Sikora A.F.M. ter Elst, Derek W. Robinson and Yueping Zhu. Second-order operators with degenerate coefficients. *Proc. London Math. Soc.*, 95(2):299–328, 2007.

[3] D.G. Aronson. Bounds for the fundamental solution of a parabolic equation. *Bull. Amer. Math. Soc.*, 73:890–896, 1967.

[4] D.G. Aronson. Non-negative solutions of linear parabolic equations. *Ann. Scuola Norm. Sup. Pisa*, 22(3):607–694, 1968.

[5] D.G. Aronson. Addendum: "non-negative solutions of linear parabolic equations"(ann. scuola norm. sup. pisa (3)**22**(1968),607-694). *Ann. Scuola Norm. Sup. Pisa*, 25(3):221–228, 1971.

[6] N. Bouleau and F. Hirsch. *Dirichlet Forms and Analysis on Wiener Space*. 1991.

[7] T. Delmotte. Parabolic harnack inequality and estimates of markov chains on graphs. *Revista Matemática Iberoamericana*, 15(1):181–232, 1999.

[8] Yuri Burago Dmitri Burago and Sergei Ivanov. *A Course in Metric Geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, 2001.

[9] E.B. Fabes. Gaussian upper bounds on fundamental solutions of parabolic equations; the method of nash. In U. Mosco G. Dell'Antonio, editor, *Dirichlet Forms, Varenna, 1992*, volume 1563 of *Lecture Notes in Mathematics*, pages 1–20. Springer-Verlag, 1993.

[10] A. Grigor'yan and L. Saloff-Coste. Stability results for harnack inequality. *Ann. Inst. Fourier, Grenoble*, 55(3):825–890, 2005.

[11] Alexander Grigor'yan. Analytic and geometric background of recurrence and non-explosion of the brownian motion on riemannian manifolds. *Bulletin(New Series) of the American Mathematical Society*, 36(2):135–249, 1999.

[12] Alexander Grigor'yan. Estimates of heat kernels on riemannian manifolds. In B. Davies and Yu. Safarov, editors, *Spectral Theory and Geometry. ICMS Instructional Conference, Edinburgh, 1998*, volume 273 of *London Math. Soc. Lecture Notes*, pages 140–225. Cambridge Univ. Press, 1999.

[13] Alexander Grigor'yan. Estimates of heat kernels on riemannian manifolds. 1999.

[14] Alexander Grigor'yan. Heat kernels on weighted manifolds and applications. *Cont. Math*, 398:93–191, 2006.

[15] Alexander Grigor'yan. *Heat Kernel and Analysis on Manifolds*, volume 47 of *AMS.IP Studies in Advanced Mathematics*. American Mathematical Society . International Press, 2009.

[16] Pavel Gyrya and Laurent Saloff-Coste. *Neumann and Dirichlet Heat Kernels in Inner Uniform Domains (Asterisque)*. American Mathematical Society, 2011.

[17] Pavel Tarasovish Gyrya. *Heat Kernel Estimates for Inner Uniform Subsets of Harnack-type Dirichlet Spaces*. PhD thesis, Cornell University, Ithaca, NY, USA, 2007.

[18] Juha Heinonen. *Lectures on Analysis on Metric Spaces*. Universitext. Springer, 2001.

[19] Gilbert Helmberg. *Getting Acquainted with Fractals*. Walter de Gruyter, 2007.

[20] Francis Hirsch. Intrinsic metrics and lipschitz functions. *Journal of Evolution Equations*, 3:11–25, 2003.

[21] Moritz Kassman. Harnack inequalities: an introduction. *Bound. Value Probl.*, 2007. Art. ID 81415,21.

[22] Jun Kigami. *Volume Doubling Measures and Heat Kernel Estimates on Self-Similar Sets*, volume 199 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 2009.

[23] P. Li and S.T. Yau. On the parabolic kernel of the schrodinger operator. *Acta. Math*, 156:153–201, 1986.

[24] Peter Li. Lecture notes on geometric analysis, 1996.

[25] Jouni Luukkainen and Eero Saksman. Every complete doubling metric space carries a doubling measure. *Proceedings of the American Mathematical Society*, 126(2):531–534, 1998.

[26] Yoichi Oshima Masayoshi Fukushima and Masayoshi Takeda. *Dirichlet Forms and Symmetric Markov Processes*, volume 19 of *Studies in Mathematics*. Walter de Gruyter, 2nd edition, 2010.

[27] L. Moschini and A. Tesei. Parabolic harnack inequality for the heat equation with inverse-squre potential. *Forum Math.*, 19:407–427, 2007.

[28] J. Moser. On harnack's theorem for elliptic differential equations. *Comm. Pure Appl. Math.*, 14:577–591, 1961.

[29] J. Moser. A harnack inequality for parabolic differential equations. *Comm. Pure Appl. Math.*, 17:101–134, 1964.

[30] J. Moser. Correction to: "a harnack inequality for parabolic differential equations". *Comm. Pure Appl. Math.*, 20:231–236, 1967.

[31] L. Saloff-Coste N. Th. Varopoulos and T. Coulhon. *Analysis and Geometry on Groups*. digitally printed edition, 2008.

[32] J. Nash. Continuity of solutions of parabolic and elliptic equations. *Amer. J. of Math.*, 80:931–954, 1958.

[33] M. Reed and B. Simon. *Method of Modern Mathematical Physics I: Functional analysis*. Academic Press Inc., revised and enlarged edition, 1970.

[34] M. Reed and B. Simon. *Method of Modern Mathematical Physics II: Fourier analysis, self-adjointness*. Academic Press Inc., 1975.

[35] Derek W. Robinson and Adam Sikora. Analysis of degerate elliptic operators of grusin type. *Mathematische Zeitschrift*, 260(3):475–508, 2008.

[36] Derek W. Robinson and Adam Sikora. Degenerate elliptic operators in one dimension. *jour. Evol. Equ.*, 10:731–759, 2010.

[37] K.T. Sturm S. Ohta. Heat flow on finsler manifolds. *Communications on Pure and Applied Mathematics*, 62(10):13861433, 2009.

[38] Laurent Saloff-coste. Sobolev inequalities in familiar and unfamiliar settings. In *In S. Sobolev Centenial Volumes, (V. Mazja, Ed*.

[39] Laurent Saloff-Coste. *Aspects of Sobolev- Type Inequalities*. Number 289 in London Mathematical Society Lecture Note. first edition, 2002.

[40] Laurent Saloff-Coste. The heat kernel and its estimates. *Advanced Studies in Pure Mathematics*, pages 1–32, 2009.

[41] R. Schoen and S.-T. Yau. *Lectures on Differential Geometry*, volume I of *Conference Proceedings and Lecture Notes in Geometry and Topology*. International Press, 1994.

[42] Robert S. Strichartz. *Differential Equations on Fractals, A tutorial*. Princeton University Press, 2006.

[43] Daniel W. Strook. *Partial Differential Equations for Probabilists*, volume 112 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2008.

[44] Karl-Theodor Sturm. On the geometry defined by dirichlet forms. *Progress in Probability*, 36:231–242, 1995.

[45] K.T. Sturm. Analysis on local dirichlet spaces i: Recurrence, conservativeness and l^p-liouville properties. *J. Reine Angew. Math*, 456:173–196, 1994.

[46] K.T. Sturm. Analysis on local dirichlet spaces ii: Upper gaussian estimates for the fundamental solutions of parabolic equations. *Osaka J. Math*, 32(2):275–312, 1995.

[47] K.T. Sturm. Analysis on local dirichlet spaces iii: The parabolic harnack inequality. *J. Math. Pures Appl.*, 75(3):273–297, 1996.

[48] N. Th. Varopoulos. Analysis on lie groups. *Journal of Functional Analysis*, 76:346–410, 1988.

[49] Wolfgang Woess. *Random Walks on Infinite Graphs and Groups*, volume 138 of *Cambridge Tracts in Mathematics*. Cambridge University Press, first edition, 2000.